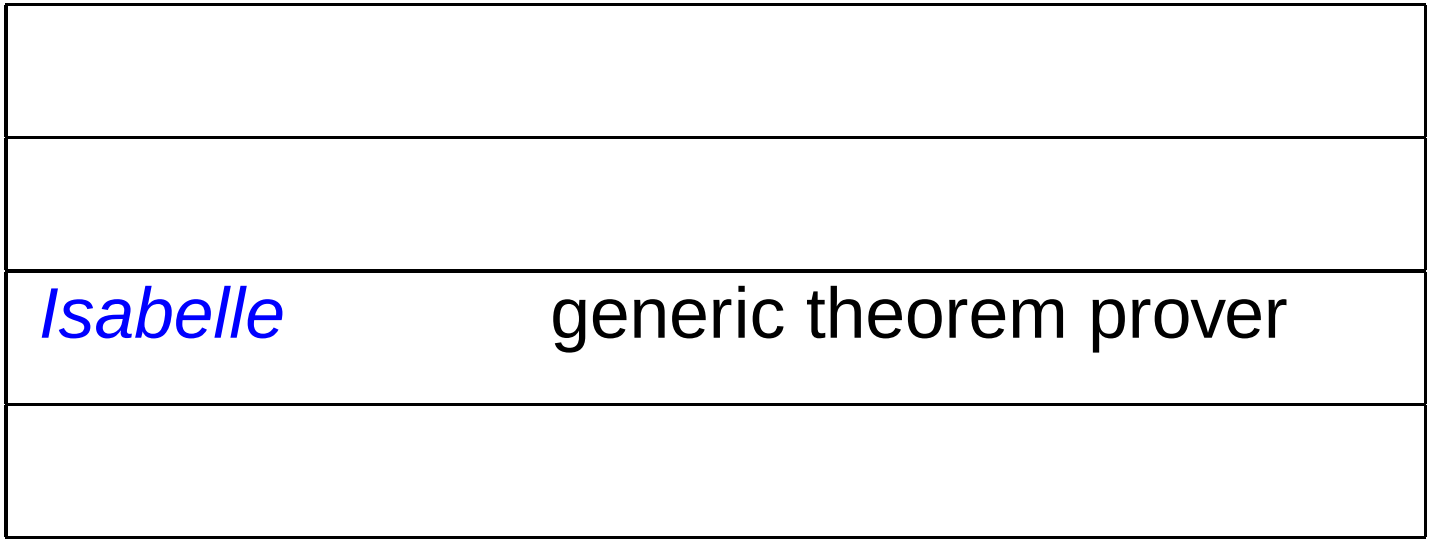
Overview of Isabelle/HOL

System Architecture



System Architecture

Isabelle	generic theorem prover
Standard ML	implementation language

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Isabelle/HOL	Isabelle instance for HOL
Isabelle	generic theorem prover
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ProofGeneral	(X)Emacs based interface
Isabelle/HOL	Isabelle instance for HOL
Isabelle	generic theorem prover
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HOL

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- logical operators ($\wedge$, $\longrightarrow$, $\forall$, $\exists$, $\ldots$)

HOL is a programming language!

Higher-order = functions are values, too!

Formulae

Syntax (in decreasing priority):

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\begin{array}{lcllll}
form & ::= & (form) & \mid & term = term & \mid \neg form \\
 & \mid & form \wedge form & \mid & form \vee form & \mid form \longrightarrow form \\
 & \mid & \forall x.\ form & \mid & \exists x.\ form &
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- $\forall x.\ \exists y.\ P\ x\ y \wedge Q\ x \equiv \forall x.\ (\exists y.\ (P\ x\ y \wedge Q\ x))$

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Hiding and renaming:

$\forall x\, y.\, (\forall x.\, P\, x\, y) \wedge Q\, x\, y \;\equiv\; \forall x_0\, y.\, (\forall x_1.\, P\, x_1\, y) \wedge G\, x_0\, y$

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  $A \wedge B \wedge C \equiv A \wedge (B \wedge C)$
- $A \longrightarrow B \longrightarrow C \equiv A \longrightarrow (B \longrightarrow C) \not\equiv (A \longrightarrow B) \longrightarrow C$ **!**

Warning

Quantifiers have low priority and need to be parenthesized:

! $P \wedge \forall x.\, Q\, x \rightsquigarrow P \wedge (\forall x.\, Q\, x)$!

Types and Terms

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Parentheses: $T1 \Rightarrow T2 \Rightarrow T3 \quad \equiv \quad T1 \Rightarrow (T2 \Rightarrow T3)$

Terms: Basic syntax

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Examples: *f (g x) y* *h (λx. f (g x))*

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Examples: $f\ (g\ x)\ y$ $h\ (\lambda x.\ f\ (g\ x))$

Parantheses: $f\ a_1\ a_2\ a_3 \equiv ((f\ a_1)\ a_2)\ a_3$

λ-calculus on one slide

Informal notation: $t[x]$

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Example: $(\lambda\ x.\ x + 5)\ 3 \quad \longrightarrow_\beta \quad (3 + 5)$

$\longrightarrow_\beta$ *in Isabelle: Don't worry, be happy*

Isabelle performs β-reduction automatically

Isabelle considers $(\lambda x.t[x])a$ and $t[a]$ equivalent

Terms and Types

Terms must be well-typed

(the argument of every function call must be of the right type)

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Notation: $t :: \tau$ means t is a well-typed term of type τ.

Type inference

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In the presence of *overloaded* functions (functions with multiple types) not always possible.

User can help with type annotations inside the term.

Example: *f (x::nat)*

Currying

Thou shalt curry your functions

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- Curried: *f* :: $\tau_1 \Rightarrow \tau_2 \Rightarrow \tau$
- Tupled: *f'* :: $\tau_1 \times \tau_2 \Rightarrow \tau$

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Advantage: *partial application* *f* a_1 with a_1 :: τ_1

Terms: Syntactic sugar

Some predefined syntactic sugar:

- *Infix:* +, -, *, #, @, . . .
- *Mixfix:* *if* _ *then* _ *else* _, *case* _ *of*, . . .

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- *Infix:* +, -, *, #, @, . . .
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Prefix binds more strongly than infix:

! $f\,x + y \equiv (f\,x) + y \not\equiv f\,(x + y)$!

Base types: bool, nat, list

Type bool

Formulae = terms of type *bool*

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True :: bool
False :: bool
$\wedge$, $\vee$, $\ldots$ *:: bool* $\Rightarrow$ *bool* $\Rightarrow$ *bool*
$\vdots$

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if-and-only-if: =

Type nat

0 :: nat
Suc :: nat $\Rightarrow$ *nat*
+, *, ... :: *nat* $\Rightarrow$ *nat* $\Rightarrow$ *nat*
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0,1,2,... :: 'a, + :: *'a* $\Rightarrow$ *'a* $\Rightarrow$ *'a*

You need type annotations: *1 :: nat, x* + *(y::nat)*

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. . . unless the context is unambiguous: *Suc z*

Type list

- *[]*: empty list
- *x # xs*: list with first element *x* ("*head*") and rest *xs* ("*tail*")
- Syntactic sugar: $[x_1,\ldots,x_n]$

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Large library:
hd, *tl*, *map*, *size*, *filter*, *set*, *nth*, *take*, *drop*, *distinct*, ...

Don't reinvent, reuse!
$\leadsto$ `HOL/List.thy`

Isabelle Theories

Theory = Module

Syntax: `theory` $MyTh$
`imports` $ImpTh_1 \ldots ImpTh_n$
`begin`
(declarations, definitions, theorems, proofs, ...)*
`end`

- $MyTh$: name of theory. Must live in file $MyTh$`.thy`
- $ImpTh_i$: name of *imported* theories. Import transitive.

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Usually:

```
theory MyTh
imports Main
⋮
```

Proof General

An Isabelle Interface

by David Aspinall

Proof General

Customized version of (x)emacs:

- all of emacs (info: `C-h i`)
- Isabelle aware (when editing `.thy` files)
- mathematical symbols (“x-symbols”)

Interaction:

- via mouse
- or keyboard (key bindings see `C-h m`)

X-Symbols

Input of funny symbols in Proof General

- via menu (“X-Symbol”)
- via ascii encoding (similar to LaTeX): `\<and>`, `\<or>`, ...
- via abbreviation: `/\`, `\/`, `-->`, ...

x-symbol	$\forall$	$\exists$	λ	$\neg$	$\wedge$	$\vee$	$\longrightarrow$	$\Rightarrow$
ascii (1)	`\<forall>`	`\<exists>`	`\<lambda>`	`\<not>`	`/\`	`\/`	`-->`	`=>`
ascii (2)	`ALL`	`EX`	`%`	`~`	`&`	`|`		

(1) is converted to x-symbol, (2) stays ascii.

Demo: terms and types