

Logic

Exercise Sheet 11

Exercise 11.1

- Show that $\Sigma \subseteq Th(Mod(\Sigma))$ holds for any set of axioms Σ . When does equality hold?
- Consider the language consisting of just the single predicate P . Show that $Cn(\forall x.P(x))$ is complete.
- Give an example for a complete theory T and a formula p (which may contain free variables), such that $T \not\models p$ and $T \not\models \neg p$.
- We have shown that a complete theory is always decidable. Give a counterexample for the reverse implication.

Exercise 11.2 Quantifier elimination for dense linear orders

Apply quantifier elimination to the following formulas:

- $\forall x. x < a \Rightarrow x < b$
- $\exists x y. \forall z. (z < x \Rightarrow z \leq y) \wedge (z > y \Rightarrow z \geq x)$
- $\exists x. \neg(\forall y. x = y \Rightarrow (y \geq a \vee y \geq b \vee y \leq c))$

Exercise 11.3 Dense linear orders with endpoints

Dense linear orders with endpoints ∞ and $-\infty$ are axiomatizable by:

$$\begin{array}{ll} \forall x y. x = y \vee x < y \vee y < x & \forall x. y. x < y \Rightarrow \exists z. x < z \wedge z < y \\ \forall x y z. x < y \wedge y < z \Rightarrow x < z & \forall x. x < \infty \vee x = \infty \\ \forall x. \neg(x < x) & \forall x. -\infty < x \vee x = -\infty \end{array}$$

Show that this theory admits quantifier elimination by describing an algorithm.

Exercise 11.4

The Compactness Theorem for first-order logic states that a countable set of formulas has a model iff all its finite subsets have a model.

Using compactness, show that if a theory is finitely axiomatizable, any countable axiomatization of it has a finite subset that axiomatizes the same theory. In other words, if $Cn(\Gamma) = Cn(\Delta)$ with Δ finite, then there is a finite $\Gamma' \subseteq \Gamma$ with $Cn(\Gamma') = Cn(\Gamma)$.