

Logic

Exercise Sheet 12

Exercise 12.1 Quantifier Elimination for Presburger Arithmetic

- a) $\forall y. 3 < x + 2y \vee 2x + y < 3$
- b) $\forall x. (\exists y. x = 2y) \Rightarrow 2x \geq 0 \vee 3x < 2$

Exercise 12.2 Quantifier Elimination for Linear Real Arithmetic

- a) $\exists xy. 2x + 3y = 7 \wedge x < y \wedge 0 < x$
- b) $\exists xy. 3x + 3y < 8 \wedge 8 < 3x + 2y$

Exercise 12.3 Definable Sets

A set $D \subseteq \mathbb{Z}$ is said to be *eventually periodic* iff there are positive numbers n and k such that for all $x \geq n$, we have $x \in D \Leftrightarrow x + k \in D$.

We say that a set $D \subseteq \mathbb{Z}$ is *definable in Presburger arithmetic*, if $D = \{x \mid p\}$ where p is a formula in Presburger arithmetic whose only free variable is x .

- a) Show that a set of integers is definable in the language of Presburger arithmetic if and only if it is eventually periodic.
- b) Show that there is no formula $p(x, y, z)$ in Presburger arithmetic such that for all $m, n, k \in \mathbb{Z}$,

$$mn = k \quad \text{iff} \quad \mathbb{Z} \models p(m, n, k)$$

Hint: Consider the set $\{x^2 \mid x \in \mathbb{Z}\}$.

Exercise 12.4 Characterizing the real numbers

- a) Find a complete axiomatization of the theory of the real numbers with addition, i.e., a set of axioms Σ , such that $\text{Cn}(\Sigma) = \text{Th}(\mathbb{R}, <, +, -, 0)$.
- b) Give a sketch of the proof that your axiomatization is complete.