

# Logic

## Exercise Sheet 4

### Exercise 4.1 Sequent Calculus Proofs

Try to find proof trees in the Sequent Calculus for each of the the following formulas. One of the formulas is not provable. Can you read off a falsifying valuation from the failed proof attempt?

- $(A \vee B) \vee C \Leftrightarrow A \vee (B \vee C)$
- $(A \Rightarrow \perp) \Leftrightarrow \neg A$
- $(A \Rightarrow B \Rightarrow C) \Rightarrow B \wedge A \vee \neg C$
- $((P \Rightarrow Q) \Rightarrow P) \Rightarrow P$

### Exercise 4.2 Weakening

We can extend the Sequent Calculus by additional *weakening rules*:

$$\frac{\Gamma \vdash \Delta}{\Gamma, p \vdash \Delta} W_L \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, p} W_R$$

Prove that adding these rules is a conservative extension, by showing that any proof in  $SC + W_L + W_R$  can be transformed into a proof in plain SC.

### Exercise 4.3 Alternative $\wedge L$ rules

Consider the following two rules:

$$\frac{\Gamma, p \vdash \Delta}{\Gamma, p \wedge q \vdash \Delta} \wedge L'_1 \qquad \frac{\Gamma, q \vdash \Delta}{\Gamma, p \wedge q \vdash \Delta} \wedge L'_2$$

Prove that replacing  $\wedge L$  by  $\wedge L'_1$  and  $\wedge L'_2$  yields an equivalent calculus, i.e., a calculus that can prove the same sequents.

### Exercise 4.4 Efficiency of SC

- Find a tautology  $p$ , such that any proof tree for  $\vdash p$  has a size at least quadratic in the size of the formula. How does resolution compare?
- The *pigeon hole principle* is the following true statement  
 If we put  $k + 1$  pigeons in  $k$  holes, then at least one hole must contain two pigeons.

For a given  $k$ , encode the statement formally in a propositional formula  $P_k$ .

Try to prove  $P_3$  in SC. What can you say about the size of proof trees for  $P_k$ ?

## Proof Rules of the Sequent Calculus:

### Axioms

$$\frac{}{\Gamma, p \vdash \Delta, p} Ax$$

$$\frac{}{\Gamma, \perp \vdash \Delta} \perp L$$

$$\frac{}{\Gamma \vdash \Delta, \top} \top R$$

### Rules

$$\frac{\Gamma \vdash \Delta, p}{\Gamma, \neg p \vdash \Delta} \neg L$$

$$\frac{\Gamma, p \vdash \Delta}{\Gamma \vdash \Delta, \neg p} \neg R$$

$$\frac{\Gamma, p, q \vdash \Delta}{\Gamma, p \wedge q \vdash \Delta} \wedge L$$

$$\frac{\Gamma \vdash \Delta, p \quad \Gamma \vdash \Delta, q}{\Gamma \vdash \Delta, p \wedge q} \wedge R$$

$$\frac{\Gamma, p \vdash \Delta \quad \Gamma, q \vdash \Delta}{\Gamma, p \vee q \vdash \Delta} \vee L$$

$$\frac{\Gamma \vdash \Delta, p, q}{\Gamma \vdash \Delta, p \vee q} \vee R$$

$$\frac{\Gamma \vdash \Delta, p \quad \Gamma, q \vdash \Delta}{\Gamma, p \Rightarrow q \vdash \Delta} \Rightarrow L$$

$$\frac{\Gamma, p \vdash \Delta, q}{\Gamma \vdash \Delta, p \Rightarrow q} \Rightarrow R$$