

Logic

Exercise Sheet 5

Exercise 5.1 More Sequent Calculus

Prove that if the sequent $\Gamma \vdash \Delta$ is derivable in the Sequent Calculus, then so is $\Gamma[x \mapsto p] \vdash \Delta[x \mapsto p]$, where $\Gamma[x \mapsto p]$ denotes the set of formulas obtained by replacing all occurrences of x in Γ by p .

Give two different proofs:

- a) one using a transformation of proof trees,
- b) and one using soundness and completeness of SC.

Exercise 5.2

Use the compactness theorem to solve Exercise 2.20 from the book (p. 117).

Exercise 5.3 Subformula property

A rule has the *subformula property* iff all formulas in the premises are subformulas of the conclusion.

- a) Check if all rules of the Sequent Calculus have the subformula property. Why is it important?
- b) Find suitable sequent rules for the connectives $\bar{\wedge}$ (“nand”) and \otimes (“xor”). Make sure they have the subformula property.

Proof Rules of the Sequent Calculus:

Axioms

$$\frac{}{\Gamma, p \vdash \Delta, p} Ax$$

$$\frac{}{\Gamma, \perp \vdash \Delta} \perp L$$

$$\frac{}{\Gamma \vdash \Delta, \top} \top R$$

Rules

$$\frac{\Gamma \vdash \Delta, p}{\Gamma, \neg p \vdash \Delta} \neg L$$

$$\frac{\Gamma, p \vdash \Delta}{\Gamma \vdash \Delta, \neg p} \neg R$$

$$\frac{\Gamma, p, q \vdash \Delta}{\Gamma, p \wedge q \vdash \Delta} \wedge L$$

$$\frac{\Gamma \vdash \Delta, p \quad \Gamma \vdash \Delta, q}{\Gamma \vdash \Delta, p \wedge q} \wedge R$$

$$\frac{\Gamma, p \vdash \Delta \quad \Gamma, q \vdash \Delta}{\Gamma, p \vee q \vdash \Delta} \vee L$$

$$\frac{\Gamma \vdash \Delta, p, q}{\Gamma \vdash \Delta, p \vee q} \vee R$$

$$\frac{\Gamma \vdash \Delta, p \quad \Gamma, q \vdash \Delta}{\Gamma, p \Rightarrow q \vdash \Delta} \Rightarrow L$$

$$\frac{\Gamma, p \vdash \Delta, q}{\Gamma \vdash \Delta, p \Rightarrow q} \Rightarrow R$$