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Logic

Exercise Sheet 7

Exercise 7.1 Characterizing models with formulas

Find a satisfiable formula p, such that $M \vDash p$ implies

a) $|D_M| \geq 2$

b) $|D_M| \geq 3$

c) D_M is infinite.

Show that any formula that has a model of size n (that is, $|D_M| = n$,) also has a model of size $n + 1$.

Exercise 7.2 Composition and substitution

The following statement about composition and substitution into terms is not well-formed:

$$\texttt{tsubst}\ i \circ \texttt{tsubst}\ j\ = \texttt{tsubst}\ (i \circ j)$$

What is the problem? (Hint: Consider the types of i and j.)

Correct the statement and then prove it by induction.

Exercise 7.3 Skolemization

Skolemize the following formulas, putting them into prenex normal form if necessary.

a) $\forall x.\exists y.\forall z.\exists w.\neg P(a, w) \vee Q(f(x), y)$

b) $(\forall x.\exists y.P(x, g(y, f(x)))) \vee \neg Q(z) \vee \neg(\forall x.R(x, y))$

Exercise 7.4

Consider the following alternative definition for `substq`, the quantifier-part of substitution:

```
let substq i quant x p =
  let x' = if mem x (fv p)
           then variant x (fv p) else x in
  quant x' (subst ((x |-> Var x') i) p);;
```

Find a counterexample, where this function leads to a variable capture in a substitution.

Exercise 7.5 Nameless representation of bound variables

a) Two formulas are called α-equivalent, if they differ only in the naming of bound variables.

   Write an OCaml function that checks if two formulas are α-equivalent.

Using an alternative representation for bound variables, we can enforce that α-equivalent formulas have the same representation:

Instead of giving names to bound variables, we write them uniformly as b_i, where i is a non-negative integer called a *deBruijn-index*. The index serves as a relative address that tells us how many quantifiers we have to skip (when moving up the term structure), until we hit the quantifier that binds the variable. The quantifiers themselves no longer carry a variable name.

Example 1: The nameless formula $\forall\forall P(b_1, b_0)$ represents $\forall x.\exists y.P(x, y)$ or, equivalently, $\forall z.\exists w.P(z, w)$.

Example 2: The formula $\forall x.P(x) \wedge (\exists y.Q(x, y))$ is written $\forall(P(b_0) \wedge \exists(Q(b_1, b_0)))$. Note that the same variable may occur as different indices in different positions.

b) Implement the nameless representation for formulas in OCaml, and define a translation from the named to the nameless representation.

c) Implement a substitution function that works on the nameless representation.