

Logic

Exercise Sheet 9

Exercise 9.1 Resolution

Refute the following formula by resolution:

$$(\forall y.Q(f(a), f(y))) \wedge (\forall x y.Q(y, f(y)) \Rightarrow P(f(x), g(y, b))) \Rightarrow (\exists x.\exists y.\exists z.P(x, y) \wedge P(f(a), g(x, b)) \wedge Q(x, z))$$

Exercise 9.2 Resolution clauses

For a set C of clauses, the set $Res(C)$ is defined as the set of all clauses which can be derived from C by resolution. Find a finite clause set C such that $Res(C)$ is infinite.

Exercise 9.3 First-order Sequent Calculus

Use the sequent calculus of first-order logic to prove the following formulas:

- a) $(\forall x.P \vee Q(x)) \Rightarrow (P \vee \forall x.Q(x))$
- b) $\neg(\forall x.\exists y.\forall z.\neg P(x, z) \wedge P(z, y))$

Exercise 9.4 Resolution and Equality

- a) A monoid is an algebraic structure consisting of a set together with a binary, associative operation $+$ and a distinguished neutral element 0 . These properties can be described by the following axioms:

$$\begin{aligned}\forall x. 0 + x &= x \\ \forall x. x + 0 &= x \\ \forall x.\forall y.\forall z. x + (y + z) &= (x + y) + z\end{aligned}$$

Prove by resolution that these axioms imply $\forall x. 0 + (x + 0) = x$ in all normal models.

- b) A group is a monoid with inverse elements, i.e., additional to the monoid axioms, we have the following properties:

$$\begin{aligned}\forall x.\exists y. x + y &= 0 \\ \forall x.\exists y. y + x &= 0\end{aligned}$$

Prove by resolution that every element of a group has exactly one inverse.

Proof Rules of the Sequent Calculus:

Axioms

$$\frac{}{\Gamma, p \vdash \Delta, p} Ax \qquad \frac{}{\Gamma, \perp \vdash \Delta} \perp L \qquad \frac{}{\Gamma \vdash \Delta, \top} \top R$$

Rules

$$\frac{\Gamma \vdash \Delta, p}{\Gamma, \neg p \vdash \Delta} \neg L \qquad \frac{\Gamma, p \vdash \Delta}{\Gamma \vdash \Delta, \neg p} \neg R$$

$$\frac{\Gamma, p, q \vdash \Delta}{\Gamma, p \wedge q \vdash \Delta} \wedge L \qquad \frac{\Gamma \vdash \Delta, p \quad \Gamma \vdash \Delta, q}{\Gamma \vdash \Delta, p \wedge q} \wedge R$$

$$\frac{\Gamma, p \vdash \Delta \quad \Gamma, q \vdash \Delta}{\Gamma, p \vee q \vdash \Delta} \vee L \qquad \frac{\Gamma \vdash \Delta, p, q}{\Gamma \vdash \Delta, p \vee q} \vee R$$

$$\frac{\Gamma \vdash \Delta, p \quad \Gamma, q \vdash \Delta}{\Gamma, p \Rightarrow q \vdash \Delta} \Rightarrow L \qquad \frac{\Gamma, p \vdash \Delta, q}{\Gamma \vdash \Delta, p \Rightarrow q} \Rightarrow R$$

$$\frac{\Gamma, \forall x.p, p[t/x] \vdash \Delta}{\Gamma, \forall x.p \vdash \Delta} \forall L \qquad \frac{\Gamma \vdash \Delta, p[y/x]}{\Gamma \vdash \Delta, \forall x.p} \forall R$$

$y \notin FV(\Gamma \cup \Delta \cup (\forall x.p))$

$$\frac{\Gamma, p[y/x] \vdash \Delta}{\Gamma, \exists x.p \vdash \Delta} \exists L \qquad \frac{\Gamma \vdash \Delta, \exists x.p, p[t/x]}{\Gamma \vdash \Delta, \exists x.p} \exists R$$

$y \notin FV(\Gamma \cup \Delta \cup (\exists x.p))$