

# Logic

## Exercise Sheet 10

### Exercise 10.1 Warm-up: Miniscoping

Transform the following formulas such that they fall into the AE-fragment:

- a)  $\exists x.\forall y.P(x) \vee (Q(y) \wedge \neg Q(x))$   
 b)  $\forall x.\exists y.\forall z.P(z) \wedge P(y) \vee \neg Q(z) \wedge Q(x)$

### Solution

a)

$$\begin{aligned}
 & \exists x.\forall y.P(x) \vee (Q(y) \wedge \neg Q(x)) \\
 \equiv & \exists x.P(x) \vee ((\forall y.Q(y)) \wedge \neg Q(x)) && \text{(push } \forall \text{ in)} \\
 \equiv & (\exists x.P(x)) \vee ((\forall y.Q(y)) \wedge (\exists x.\neg Q(x))) && \text{(push } \exists \text{ in)} \\
 \equiv & \forall y.(\exists x.P(x) \vee (Q(y) \wedge (\exists x.\neg Q(x)))) && \text{(pull } \forall \text{ out)} \\
 \equiv & \forall y.\exists x.P(x) \vee (Q(y) \wedge \neg Q(x)) && \text{(pull } \exists \text{ out)}
 \end{aligned}$$

b)

$$\begin{aligned}
 & \forall x.\exists y.\forall z.P(z) \wedge P(y) \vee \neg Q(z) \wedge Q(x) \\
 \equiv & \forall x.\exists y.\forall z.(P(z) \vee \neg Q(z)) \wedge (P(z) \vee Q(x)) \wedge (P(y) \vee \neg Q(z)) \wedge (P(y) \vee Q(x)) && \text{(put into CNF)} \\
 \equiv & \forall x.\exists y.(\forall z.P(z) \vee \neg Q(z)) \wedge (\forall z.P(z) \vee Q(x)) \wedge (P(y) \vee \forall z.\neg Q(z)) \wedge (P(y) \vee Q(x)) && \text{(push } \forall \text{ in)} \\
 \equiv & \forall x.(\forall z.P(z) \vee \neg Q(z)) \wedge (\forall z.P(z) \vee Q(x)) \wedge (\exists y.(P(y) \vee \forall z.\neg Q(z)) \wedge (P(y) \vee Q(x))) && \text{(partly push } \exists \text{ in)} \\
 \equiv & \forall x.(\forall z.P(z) \vee \neg Q(z)) \wedge (\forall z.P(z) \vee Q(x)) \wedge (\exists y.(P(y) \vee (\forall z.\neg Q(z) \wedge Q(x)))) && \text{(factor out } P(y)) \\
 \equiv & \forall x.(\forall z.P(z) \vee \neg Q(z)) \wedge (\forall z.P(z) \vee Q(x)) \wedge ((\exists y.P(y)) \vee (\forall z.\neg Q(z) \wedge Q(x))) && \text{(push } \exists \text{ in further)} \\
 \equiv & \forall xz.\exists y.(P(z) \vee \neg Q(z)) \wedge (P(z) \vee Q(x)) \wedge (P(y) \vee (\neg Q(z) \wedge Q(x))) && \text{(pull } \forall, \text{ then } \exists)
 \end{aligned}$$

### Exercise 10.2 FOL with only one variable

We consider formulas without function symbols, where the only variable name that is allowed is  $x$ . (Example:  $\forall x.\exists x.P(x, x) \vee Q(x)$ ).

Show that this class is decidable.

## Solution

The given class is decidable because, like for the monadic fragment, every formula can be rewritten such that it falls into the AE-fragment.

The proof is a modified version of the proof for the monadic fragment. Recall that we defined the function *miniscope*, which repeatedly pushes quantifiers inside the formula, such that they are no longer nested. The same works for the new class, and we only need to adapt the proof of the following lemma:

**Lemma 1.** *Assume that  $p$  contains just a single variable  $x$ . If  $p$  is flat, then so is  $\text{push}_x x p$ .*

*Proof.* If  $x \notin FV(p)$ , then  $\text{push}_x x p = p$  which is flat by assumption.

If  $x \in FV(p)$ , then  $\text{push}_x$  builds a DNF  $C_1 \vee \dots \vee C_n$ , where for each  $i$  we have  $C_i^+ = \{q \in C_i \mid x \in FV(q)\}$  and  $C_i^- = \{q \in C_i \mid x \notin FV(q)\}$ .

Now, all  $q \in C_i$  are flat by assumption, hence  $C_i^-$  is flat. Every  $q \in C_i^+$  must be of the form  $P(x, \dots, x)$ , since it cannot be quantified (because  $x \in FV(q)$ ) or a conjunction/disjunction (because we already have a DNF). Hence  $FV(q) = \{x\}$ , and hence  $\exists x.C_i^+$  is flat, and so is the result of *miniscope*.  $\square$

## Exercise 10.3 Finite models for EA

Consider a formula in the EA-Fragment, i.e., of the form  $p = \exists x_1, \dots, x_n. \forall y_1, \dots, y_m. q$ , where  $q$  is quantifier-free and without function symbols.

Show that  $p$  has a model iff it has a finite model of size  $n$  (or 1 in the case  $n = 0$ ).

## Solution

If  $p$  has a finite model then it surely has a model. For the reverse implication, assume that  $\models_M p$ . By interpreting the formula, this means that there exist  $a_1, \dots, a_n \in D_M$ , such that for all  $b_1, \dots, b_m \in D_M$ , the property specified by  $q$  holds.

In particular, it holds for all  $b_1, \dots, b_m \in \{a_1, \dots, a_n\}$ , and hence we can restrict the model  $M$  to  $M'$ , where  $D_{M'} = \{a_1, \dots, a_n\}$ , and predicate symbols are interpreted as in  $M$ .

The size of  $|D_{M'}| \leq n$ . It is not necessarily equal to  $n$ , since some  $a_i, a_j$  may coincide. However, we can always add dummy elements to the model as we did in Ex. 7.1, to obtain a model of size  $n$ .

## Exercise 10.4 Strictly positive formulas

Find a simple procedure for deciding validity of formulas built only from  $\forall, \exists, \wedge, \vee, \top, \perp$ , where function symbols are allowed.

**Solution**

Consider the model  $M$  with  $D_M = \{*\}$ , where all predicate symbols are interpreted as being false. It is easily shown by induction that formulas consisting only of  $\forall, \exists, \wedge, \vee, \perp$  are never satisfied by this model, regardless of the valuation. The only way of producing a valid formula is thus by the  $\top$  symbol.

Hence, a simple decision procedure for this class of formulas is to eliminate  $\top$  using the function *simplify*. The resulting formula is either equal to  $\top$ , or no longer contains  $\top$  and is thus invalid.

**Exercise 10.5**

Implement the decision procedure you developed in Ex. 10.2.

**Solution**

The decision procedure is the same as for the monadic fragment, which is called *wang* in the book.