

Logic

Exercise Sheet 13

Exercise 13.1 $\text{Th}(\mathbb{R}, 0, 1, +, -, \min, \max, <, \leq, =)$

- a) Show that quantifier elimination for real linear arithmetic still works if we add the binary functions \min and \max .
 b) Use the elimination to prove the following formula:

$$\forall abcd. \max(a, b) < \min(c, d) \Rightarrow ((\forall x. x > 2a \vee x \leq a) \vee (\exists y. 0 \leq y \wedge \max(c, y) = c))$$

- c) Show that we can also add the function $|\cdot|$ that yields the absolute value of its argument.

Solution

- a) We first rewrite the terms with $\max(s, t) + u = \max(s + u, t + u)$ (and similar for \min) to pull \max and \min out. Then we can use the following equivalences to eliminate \max :

$$\begin{aligned} \max(s, t) < u &\equiv s < u \wedge t < u \\ u < \max(s, t) &\equiv u < s \vee u < t \\ u = \max(s, t) &\equiv u = s \wedge t \leq s \vee u = t \wedge s \leq t \end{aligned}$$

For \min and for \leq , the rules are similar.

b)

$$p = \forall abcd. \max(a, b) < \min(c, d) \Rightarrow \underbrace{((\forall x. x > 2a \vee x \leq a))}_{p_1} \vee \underbrace{(\exists y. 0 \leq y \wedge \max(c, y) = c)}_{p_2}$$

$$\begin{aligned} p_1 &\equiv \neg \exists x. x \leq 2a \wedge x > a \\ &\equiv \neg(a < 2a) \\ &\equiv \neg(0 < a) \\ &\equiv a \leq 0 \end{aligned}$$

$$\begin{aligned} p_2 &\equiv \exists y. 0 \leq y \wedge (c = c \wedge y \leq c \vee c = y \wedge c \leq y) \\ &\equiv \exists y. (0 \leq y \wedge y \leq c) \vee (0 \leq y \wedge c = y \wedge c \leq y) \\ &\equiv (\exists y. 0 \leq y \wedge y \leq c) \vee (\exists y. 0 \leq y \wedge c = y \wedge c \leq y) \\ &\equiv (0 \leq c) \vee (0 \leq c) \\ &\equiv 0 \leq c \end{aligned}$$

$$\begin{aligned}
p &\equiv \neg \exists abcd. \max(a, b) < \min(c, d) \wedge \neg(p_1 \vee p_2) \\
&\equiv \neg \exists abcd. \max(a, b) < \min(c, d) \wedge 0 < a \wedge c < 0 \\
&\equiv \neg \exists abcd. \max(a, b) < c \wedge \max(a, b) < d \wedge 0 < a \wedge c < 0 \\
&\equiv \neg \exists abcd. \underline{a < c} \wedge b < c \wedge a < d \wedge b < d \wedge \underline{0 < a} \wedge \underline{c < 0} \\
&\equiv \neg \exists abcd. \perp \\
&\equiv \top
\end{aligned}$$

c) The absolute value can be eliminated by $|t| = \max(t, -t)$.

Exercise 13.2 Induction

Use induction to show that there is no quantifier-free formula with just $\wedge, \vee, \leq, =$ that is equivalent to $x < y$.

Solution

It suffices to show that there is no formula with a single variable x that is equivalent to $x < 0$.

We recall from analysis that a set S is called *open* if every point has a neighbourhood contained in S , i.e., $\forall x \in S. \exists \varepsilon > 0. \forall y. |x + y| < \varepsilon \Rightarrow y \in S$. Complements of open sets are called *closed*. If two sets are closed, then so is their union and intersection.

We show by induction that the set $\{x \mid p\}$ is closed for every p with free variable x : Clearly, the sets defined by $x = c$, $x \leq c$ and $c \leq x$ are all closed. Since closedness is preserved by intersections and (binary) unions, the whole formula also defines a closed set.

However, the set $\{x \mid x < 0\}$ is not closed, and hence not definable by this class of formulas. Of course the picture changes if we allow negation, since $x < y \equiv \neg(y \leq x)$.

Exercise 13.3 Independence

A set Γ of propositional formulas is called *independent* if $p \in \Gamma$ implies $\Gamma \setminus \{p\} \not\models p$.

Show that any finite set has an equivalent independent subset.

Solution

Assume that Γ is finite. Hence we can use induction (over its size): If Γ is independent, then we are done. Otherwise there is a formula $p \in \Gamma$, such that $\Gamma \setminus \{p\} \models p$. Then Γ is equivalent to $\Gamma \setminus \{p\}$ which is a smaller set. By induction hypothesis, $\Gamma \setminus \{p\}$ has an equivalent independent subset.

Exercise 13.4 Brainstorming

Collect about 30 important pieces of terminology that were used in the lecture and that you want to remember for the exam.

Also collect the three results that you find most interesting.