

Logic

Exercise Sheet 4

Exercise 4.1 Sequent Calculus Proofs

Try to find proof trees in the Sequent Calculus for each of the the following formulas. One of the formulas is not provable. Can you read off a falsifying valuation from the failed proof attempt?

- a) $(A \vee B) \vee C \Leftrightarrow A \vee (B \vee C)$
- b) $(A \Rightarrow \perp) \Leftrightarrow \neg A$
- c) $(A \Rightarrow B \Rightarrow C) \Rightarrow B \wedge A \vee \neg C$
- d) $((P \Rightarrow Q) \Rightarrow P) \Rightarrow P$

Solution

a) We only show one direction of the equivalence:

$$\begin{array}{c}
 \frac{}{A \vdash A, B, C} \text{Ax} \quad \frac{}{B \vdash A, B, C} \text{Ax} \\
 \hline
 \frac{}{A \vee B \vdash A, B, C} \quad \frac{}{C \vdash A, B, C} \text{Ax} \\
 \hline
 \frac{}{(A \vee B) \vee C \vdash A, B, C} \vee L \\
 \hline
 \frac{}{(A \vee B) \vee C \vdash A, B \vee C} \vee R \\
 \hline
 \frac{}{(A \vee B) \vee C \vdash A \vee (B \vee C)} \vee R \\
 \hline
 \frac{}{\vdash (A \vee B) \vee C \Rightarrow A \vee (B \vee C)} \Rightarrow R
 \end{array}$$

b)

$$\begin{array}{c}
 \frac{}{A \vdash A} \text{Ax} \quad \frac{}{\perp \vdash A} \perp L \\
 \hline
 \frac{}{A \Rightarrow \perp, A \vdash \perp} \Rightarrow L \\
 \hline
 \frac{}{A \Rightarrow \perp \vdash \neg A} \neg R \\
 \hline
 \frac{}{\vdash (A \Rightarrow \perp) \Rightarrow \neg A} \Rightarrow R \\
 \hline
 \frac{}{\vdash (A \Rightarrow \perp) \Rightarrow \neg A} \Rightarrow R \quad \frac{}{A \vdash \perp, A} \text{Ax} \\
 \hline
 \frac{}{\vdash \neg A \Rightarrow (A \Rightarrow \perp)} \neg R \\
 \hline
 \frac{}{\vdash \neg A \Rightarrow (A \Rightarrow \perp)} \Rightarrow R \\
 \hline
 \frac{}{\vdash ((A \Rightarrow \perp) \Rightarrow \neg A) \wedge (\neg A \Rightarrow (A \Rightarrow \perp))} \wedge R
 \end{array}$$

c)

$$\begin{array}{c}
\frac{C \vdash B, A \quad C \vdash A}{C \vdash B \wedge A, A} \wedge R \quad B \Rightarrow C, C \vdash B \wedge A \\
\hline
A \Rightarrow B \Rightarrow C, C \vdash B \wedge A \quad \Rightarrow L \\
\hline
A \Rightarrow B \Rightarrow C \vdash B \wedge A, \neg C \quad \neg R \\
\hline
A \Rightarrow B \Rightarrow C \vdash B \wedge A \vee \neg C \quad \vee R \\
\hline
\vdash (A \Rightarrow B \Rightarrow C) \Rightarrow B \wedge A \vee \neg C \quad \Rightarrow R
\end{array}$$

The top-most, left-most leaf of the proof attempt reads $C \vdash B, A$. The proof rules of the sequent calculus would allow a valid deduction if $C = \perp$ ($\perp L$) or $B = \top$ ($\top R$) or $A = \top$ ($\top R$). Assuming the opposite leads to the falsifying valuation $A = \text{false}, B = \text{false}, C = \text{true}$. In general, a falsifying valuation can be found by mapping the atoms left of the \vdash to true and the others to false.

d)

$$\begin{array}{c}
\frac{\overline{P \vdash Q, P} \quad Ax}{\vdash P \Rightarrow Q, P} \Rightarrow R \quad \frac{\overline{P \vdash P} \quad Ax}{(P \Rightarrow Q) \Rightarrow P \vdash P} \Rightarrow L \\
\hline
\vdash ((P \Rightarrow Q) \Rightarrow P) \Rightarrow P \quad \Rightarrow R
\end{array}$$

Exercise 4.2 Weakening

We can extend the Sequent Calculus by additional *weakening rules*:

$$\frac{\Gamma \vdash \Delta}{\Gamma, p \vdash \Delta} W_L \quad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, p} W_R$$

Prove that adding these rules is a conservative extension, by showing that any proof in $SC + W_L + W_R$ can be transformed into a proof in plain SC.

Solution

The proof proceeds in two steps. We show that adding propositional variables to either side of any sequent preserves provability. Then we can use this transformation to remove all occurrences of W_L and W_R in a proof.

Assume that $\Gamma \vdash \Delta$ is provable in the plain sequent calculus. By induction over the structure of the proof tree, we show that $\Gamma, p \vdash \Delta$ is also provable. We have to consider all axioms and rules.

- a) *Axioms*: Assume that $\Gamma \vdash \Delta$ was derived using Ax . Then $\Gamma = \Gamma', q$ and $\Delta = \Delta', q$. Then, $\Gamma, p \vdash \Delta$ is also derivable using Ax :

$$\frac{\overline{\Gamma', q, p \vdash \Delta', q} \quad Ax}{\Gamma', q, p \vdash \Delta', q}$$

For the other axioms we argue similarly.

- b) *Rules*: Assume that the proof for $\Gamma \vdash \Delta$ starts with the application of $\wedge R$, i.e., $\Delta = \Delta', q \wedge r$ and

$$\frac{\Gamma \vdash \Delta', q \quad \Gamma \vdash \Delta', r}{\Gamma \vdash \Delta', q \wedge r} \wedge R.$$

By induction hypothesis we know that $\Gamma, p \vdash \Delta', q$ and $\Gamma, p \vdash \Delta', r$ are provable. Thus

$$\frac{\Gamma, p \vdash \Delta', q \quad \Gamma, p \vdash \Delta', r}{\Gamma, p \vdash \Delta', q \wedge r} \wedge R$$

yields a proof of $\Gamma, p \vdash \Delta$.

For the other rules we argue similarly, showing that every rule allows adding p at the left.

Analogously, we can show that if $\Gamma \vdash \Delta$ is provable then so is $\Gamma \vdash \Delta, p$.

Now assume a proof in the extended calculus $SC + W_L + W_R$. We pick arbitrary steps where W_L and W_R are applied:

$$\frac{\Gamma_1 \vdash \Delta_1}{\Gamma_1, p \vdash \Delta_1} W_L \qquad \frac{\Gamma_2 \vdash \Delta_2}{\Gamma_2 \vdash \Delta_2, q} W_R$$

With the previous argument, we know that the proofs of $\Gamma_1 \vdash \Delta_1$ and $\Gamma_2 \vdash \Delta_2$ can be transformed to proofs of $\Gamma_1, p \vdash \Delta_1$ and $\Gamma_2 \vdash \Delta_2, q$, respectively. Hence, instead of applying W_L and W_R , we can always change the proof of their premises to a proof of their conclusions.

Exercise 4.3 Alternative $\wedge L$ rules

Consider the following two rules:

$$\frac{\Gamma, p \vdash \Delta}{\Gamma, p \wedge q \vdash \Delta} \wedge L'_1 \qquad \frac{\Gamma, q \vdash \Delta}{\Gamma, p \wedge q \vdash \Delta} \wedge L'_2$$

Prove that replacing $\wedge L$ by $\wedge L'_1$ and $\wedge L'_2$ yields an equivalent calculus, i.e., a calculus that can prove the same sequents.

Solution

- a) Assume a proof in the original calculus SC . We pick an arbitrary step where $\wedge L$ is applied:

$$\frac{\Gamma, p, q \vdash \Delta}{\Gamma, p \wedge q \vdash \Delta} \wedge L$$

Note that both sides of a sequent are sets. In particular, the formula $p \wedge q$ may be considered as the set $\{p \wedge q\}$, or, equivalently, as two formulas $p \wedge q$. Then, the step above can be transformed into

$$\frac{\Gamma, p, q \vdash \Delta}{\Gamma, p, p \wedge q \vdash \Delta} \wedge L'_2 \qquad \frac{\Gamma, p, p \wedge q \vdash \Delta}{\Gamma, p \wedge q, p \wedge q \vdash \Delta} \wedge L'_1$$

which does not use rule $\wedge L$ of the original SC.

- b) Assume a proof in the modified calculus, i.e., using the rules $\wedge L'_1$ and $\wedge L'_2$. We pick arbitrary steps where $\wedge L'_1$ and $\wedge L'_2$ are applied:

$$\frac{\Gamma, p \vdash \Delta}{\Gamma, p \wedge q \vdash \Delta} \wedge L'_1 \qquad \frac{\Gamma, q \vdash \Delta}{\Gamma, p \wedge q \vdash \Delta} \wedge L'_2$$

We can replace these steps syntactically by

$$\frac{\frac{\Gamma, p \vdash \Delta}{\Gamma, p, q \vdash \Delta} W_L}{\Gamma, p \wedge q \vdash \Delta} \wedge L \qquad \frac{\frac{\Gamma, q \vdash \Delta}{\Gamma, p, q \vdash \Delta} W_L}{\Gamma, p \wedge q \vdash \Delta} \wedge L$$

and with the result of exercise 4.2 we obtain an equivalent proof in the original sequent calculus.

Exercise 4.4 Efficiency of SC

- a) Find a tautology p , such that any proof tree for $\vdash p$ has a size at least quadratic in the size of the formula. How does resolution compare?
- b) The *pigeon hole principle* is the following true statement

If we put $k + 1$ pigeons in k holes, then at least one hole must contain two pigeons.

For a given k , encode the statement formally in a propositional formula P_k .

Try to prove P_3 in SC. What can you say about the size of proof trees for P_k ?

Solution

- a) Only the rules $\vee L$, $\Rightarrow R$, and $\wedge R$ lead to a proof explosion in the sequent calculus. Thus, for example, the tautology $p = (A \vee A \vee \dots \vee A \Rightarrow A \wedge A \wedge \dots \wedge A)$ causes a proof quadratic in the size of p . Intuitively, every A on the right hand side is deduced from every A on the left-hand side.

Resolution performs much better on p . After conversion to CNF, we obtain several clauses of the form $[\neg A]$ and $[A]$, which can be resolved to the empty clause by just one step.

- b) Let the atom $Q_{p,h}$ denote that pigeon p is in hole h . Then, the pigeon hole principle can be expressed as follows:

$$\left(\bigwedge_{0 \leq p \leq k} \bigvee_{0 \leq h < k} Q_{p,h} \right) \Rightarrow \bigvee_{1 \leq h < k} \left(\bigvee_{\substack{0 \leq p_1 \leq k \\ 0 \leq p_2 \leq k \\ p_1 \neq p_2}} (Q_{p_1,h} \wedge Q_{p_2,h}) \right)$$

The instance P_2 reads as follows:

$$(Q_{0,0} \vee Q_{0,1}) \wedge (Q_{1,0} \vee Q_{1,1}) \wedge (Q_{2,0} \vee Q_{2,1}) \Rightarrow \\ (Q_{0,0} \wedge Q_{1,0}) \vee (Q_{0,0} \wedge Q_{2,0}) \vee (Q_{1,0} \wedge Q_{2,0}) \vee (Q_{0,1} \wedge Q_{1,1}) \vee (Q_{0,1} \wedge Q_{2,1}) \vee (Q_{1,1} \wedge Q_{2,1})$$

The SC rules $\vee L$ and $\wedge R$ lead to non-linear proof trees, and the particular structure of the formula (many disjunctions in the premise, many conjunctions in the conclusion), leads to exponential proofs.

Proof Rules of the Sequent Calculus:

Axioms

$$\frac{}{\Gamma, p \vdash \Delta, p} Ax \qquad \frac{}{\Gamma, \perp \vdash \Delta} \perp L \qquad \frac{}{\Gamma \vdash \Delta, \top} \top R$$

Rules

$$\frac{\Gamma \vdash \Delta, p}{\Gamma, \neg p \vdash \Delta} \neg L \qquad \frac{\Gamma, p \vdash \Delta}{\Gamma \vdash \Delta, \neg p} \neg R$$
$$\frac{\Gamma, p, q \vdash \Delta}{\Gamma, p \wedge q \vdash \Delta} \wedge L \qquad \frac{\Gamma \vdash \Delta, p \quad \Gamma \vdash \Delta, q}{\Gamma \vdash \Delta, p \wedge q} \wedge R$$
$$\frac{\Gamma, p \vdash \Delta \quad \Gamma, q \vdash \Delta}{\Gamma, p \vee q \vdash \Delta} \vee L \qquad \frac{\Gamma \vdash \Delta, p, q}{\Gamma \vdash \Delta, p \vee q} \vee R$$
$$\frac{\Gamma \vdash \Delta, p \quad \Gamma, q \vdash \Delta}{\Gamma, p \Rightarrow q \vdash \Delta} \Rightarrow L \qquad \frac{\Gamma, p \vdash \Delta, q}{\Gamma \vdash \Delta, p \Rightarrow q} \Rightarrow R$$