

Logic

Exercise Sheet 5

Exercise 5.1 More Sequent Calculus

Prove that if the sequent $\Gamma \vdash \Delta$ is derivable in the Sequent Calculus, then so is $\Gamma[x \mapsto p] \vdash \Delta[x \mapsto p]$, where $\Gamma[x \mapsto p]$ denotes the set of formulas obtained by replacing all occurrences of x in Γ by p .

Give two different proofs:

- a) one using a transformation of proof trees,
- b) and one using soundness and completeness of SC.

Solution

- a) Given a proof of $\Gamma \vdash \Delta$, we can apply the substitution $x \mapsto p$ throughout the proof tree and obtain a valid proof of $\Gamma[x \mapsto p] \vdash \Delta[x \mapsto p]$. Formally, this is an induction on the structure of the proof. Each case in the induction considers one proof rule or axiom, and we have to show that the new proof is still an instance of the axiom. This is trivial in all cases, since the rules are given as schemas that do only concern the structure of the formulas, which is not changed by the substitution.
- b) We can also prove the claim by using soundness and completeness. Assume that $\Gamma = \{p_1, \dots, p_n\}$, $\Delta = \{q_1, \dots, q_m\}$ and $\Gamma \vdash \Delta$ is derivable. By soundness we know that $p_1 \wedge \dots \wedge p_n \Rightarrow q_1 \vee \dots \vee q_m$ is a tautology. By the substitution lemma (Corollary 2.4) we know that substituting p for x yields a tautology again, and the completeness theorem asserts that the resulting sequent is provable.

Exercise 5.2

Use the compactness theorem to solve Exercise 2.20 from the book (p. 117).

Solution

For each $b \in B$, we write G_b for the finite set of girlfriends of b . We use atom M_{bg} to denote that b marries g . We say that a set of boys is *consistent*, if every boy can marry one of his girlfriends without any of them committing bigamy.

We have to prove that B is consistent if all finite subsets of B are. To this aim, we define the set of formulas $\text{Con}(A)$ of a set $A \subseteq B$ that expresses consistency:

$$\text{Con}(A) = \left\{ \bigvee_{g \in G_b} M_{bg} \mid b \in A \right\} \\ \cup \{ \neg(M_{b_1g} \wedge M_{b_2g}) \mid b_1, b_2 \in A, g \in G_{b_1} \cap G_{b_2} \}$$

Due to compactness, $\text{Con}(B)$ is satisfiable when any finite subset of it is satisfiable. So take a finite set $F \subseteq \text{Con}(B)$, then we can take the set A of boys that is mentioned in some formula in F , and $F \subseteq \text{Con}(A)$. But $\text{Con}(A)$ is satisfiable by assumption, and hence F is satisfiable, too.

Note that when encoding some constraints into a propositional formula using the big \bigwedge and \bigvee notation, we must make sure that the sets involved are finite. For example,

$$\bigwedge_{\substack{b_1, b_2 \in B \\ g \in G_{b_1}}} \neg(M_{b_1g} \wedge M_{b_2g})$$

is not a propositional formula, since formulas must be finite.

Exercise 5.3 Subformula property

A rule has the *subformula property* iff all formulas in the premises are subformulas of the conclusion.

- Check if all rules of the Sequent Calculus have the subformula property. Why is it important?
- Find suitable sequent rules for the connectives $\bar{\wedge}$ (“nand”) and \otimes (“xor”). Make sure they have the subformula property.

Solution

- The subformula property ensures that proofs in the calculus always terminate: When searching for proofs, we cannot run in an infinite loop.

b)

$$\frac{\Gamma \vdash \Delta, p \quad \Gamma \vdash \Delta, q}{\Gamma, p \bar{\wedge} q \vdash \Delta} \bar{\wedge}L \qquad \frac{\Gamma, p, q \vdash \Delta}{\Gamma \vdash \Delta, p \bar{\wedge} q} \bar{\wedge}R$$

$$\frac{\Gamma, p \vdash \Delta, q \quad \Gamma, q \vdash \Delta, p}{\Gamma, p \otimes q \vdash \Delta} \otimes L \qquad \frac{\Gamma \vdash \Delta, p, q \quad \Gamma, p, q \vdash \Delta}{\Gamma \vdash \Delta, p \otimes q} \otimes R$$

Proof Rules of the Sequent Calculus:

Axioms

$$\frac{}{\Gamma, p \vdash \Delta, p} Ax \qquad \frac{}{\Gamma, \perp \vdash \Delta} \perp L \qquad \frac{}{\Gamma \vdash \Delta, \top} \top R$$

Rules

$$\frac{\Gamma \vdash \Delta, p}{\Gamma, \neg p \vdash \Delta} \neg L \qquad \frac{\Gamma, p \vdash \Delta}{\Gamma \vdash \Delta, \neg p} \neg R$$
$$\frac{\Gamma, p, q \vdash \Delta}{\Gamma, p \wedge q \vdash \Delta} \wedge L \qquad \frac{\Gamma \vdash \Delta, p \quad \Gamma \vdash \Delta, q}{\Gamma \vdash \Delta, p \wedge q} \wedge R$$
$$\frac{\Gamma, p \vdash \Delta \quad \Gamma, q \vdash \Delta}{\Gamma, p \vee q \vdash \Delta} \vee L \qquad \frac{\Gamma \vdash \Delta, p, q}{\Gamma \vdash \Delta, p \vee q} \vee R$$
$$\frac{\Gamma \vdash \Delta, p \quad \Gamma, q \vdash \Delta}{\Gamma, p \Rightarrow q \vdash \Delta} \Rightarrow L \qquad \frac{\Gamma, p \vdash \Delta, q}{\Gamma \vdash \Delta, p \Rightarrow q} \Rightarrow R$$