

## Logic

### Exercise Sheet 8

#### Exercise 8.1 Warm-Up

a) For each of the following formulas, give the domain of the Herbrand model.

i)  $P(x) \Rightarrow P(c)$                       ii)  $P(x) \Rightarrow Q(f(x), g(c))$                       iii)  $\forall x. \exists y. P(x, y)$

b) Which of the following unification problems are solvable? Give the most general unifier if it exists.

i)  $f(x, y) \stackrel{?}{=} f(h(a), x)$                       iii)  $f(x, b) \stackrel{?}{=} f(h(y), z)$   
ii)  $f(x, y) \stackrel{?}{=} f(h(x), x)$                       iv)  $f(x, x) \stackrel{?}{=} f(h(y), y)$

#### Solution

- a) i)  $\{c\}$   
ii)  $\{c, f(c), g(c), f(f(c)), f(g(c)), g(f(c)), g(g(c)), f(f(f(c))), \dots\}$   
iii) We must skolemize first, and get  $P(x, s_y(x))$ , dropping the universal quantifier. Then the Herbrand universe is  $\{c, s_x(c), s_x(s_x(c)), \dots\}$ . Note that we added a constant  $c$ , since otherwise the universe would be empty.
- b) i) Solvable with mgu  $\sigma = \{x \mapsto h(a), y \mapsto h(a)\}$ .  
ii) Unsolvable: If there were a unifier  $\sigma$ , we would have  $f(\sigma(x), \sigma(y)) = f(h(\sigma(x)), \sigma(x))$ , which is impossible, since  $\sigma(x) \neq h(\sigma(x))$ .  
iii) Solvable with mgu  $\sigma = \{x \mapsto h(y), z \mapsto b\}$ .  
iv) Unsolvable: If we had a unifier  $\sigma$ , then  $f(\sigma(x), \sigma(x)) = f(h(\sigma(y)), \sigma(y))$ . This implies  $\sigma(x) = h(\sigma(y))$  and  $\sigma(x) = \sigma(y)$ , and hence  $h(\sigma(y)) = \sigma(y)$ , which is again impossible.

#### Exercise 8.2 Herbrand Model

Prove the following formula by skolemizing its negation, and finding a set of ground instances in the Herbrand model that is propositionally unsatisfiable:

$$(\forall x. P(x, f(x))) \Rightarrow (\exists y. P(c, y))$$

## Solution

We put the negation in prenex normal form and drop the universal quantifiers. Since there are no existentials, we need not skolemize:

$$\begin{aligned} & \neg((\forall x.P(x, f(x))) \Rightarrow (\exists y.P(c, y))) \\ \equiv & (\forall x.P(x, f(x))) \wedge \neg(\exists y.P(c, y)) \\ \equiv & (\forall x.P(x, f(x))) \wedge (\forall y.\neg P(c, y)) \\ \equiv & \forall xy.P(x, f(x)) \wedge \neg P(c, y) \\ \rightsquigarrow & P(x, f(x)) \wedge \neg P(c, y) \end{aligned}$$

The Herbrand universe is  $\{c, f(c), f(f(c)), \dots\}$ . With the substitution  $\{x \mapsto c, y \mapsto f(c)\}$ , we get the ground instance  $P(c, f(c)) \wedge \neg P(c, f(c))$  which is propositionally unsatisfiable. Hence the original quantified formula is unsatisfiable, and the original formula is valid.

## Exercise 8.3 Resolution

Formalize the following propositions in first-order logic and use resolution to show that a) implies b):

- a) Professor  $p$  is happy if all his students like logic.
- b) Professor  $p$  is happy if he has no students.

## Solution

We use the unary predicates  $H$  (“is happy”), and  $L$  (“likes logic”), a binary predicate  $S$  (“is student of”) and a constant  $p$ .

For a), we get the formula  $p_a = (\forall x.S(x, p) \Rightarrow L(x)) \Rightarrow H(p)$ .

For b), we get the formula  $p_b = (\forall x.\neg S(x, p)) \Rightarrow H(p)$ .

To prove  $p_a \Rightarrow p_b$  by resolution, we skolemize the negation and put it in conjunctive normal form:

$$\begin{aligned} & \neg(p_a \Rightarrow p_b) && \text{(negate)} \\ \equiv & p_a \wedge \neg p_b && \text{(pull in } \neg) \\ \equiv & p_a \wedge (\forall x.\neg S(x, p)) \wedge \neg H(p) \\ \equiv & ((\exists x.S(x, p) \wedge \neg L(x)) \vee H(p)) \wedge (\forall x.\neg S(x, p)) \wedge \neg H(p) \\ \equiv & (\exists x.(S(x, p) \vee H(p)) \wedge (\neg L(x) \vee H(p))) \wedge (\forall x.\neg S(x, p)) \wedge \neg H(p) \\ \rightsquigarrow & (S(c_x, p) \vee H(p)) \wedge (\neg L(c_x) \vee H(p)) \wedge \neg S(x, p) \wedge \neg H(p) && \text{(skolemize and drop } \forall) \\ \rightsquigarrow & \{S(c_x, p), H(p)\}, \quad \{\neg L(c_x), H(p)\}, \quad \{\neg S(x, p)\}, \quad \{\neg H(p)\} && \text{(clause form)} \end{aligned}$$

Now we can resolve the first and the third clause, instantiating with  $\{x \mapsto c_x\}$ . This gives  $\{H(p)\}$ . Resolving with the last clause yields the empty clause  $\{\}$ . This proves that  $\neg(p_a \Rightarrow p_b)$  is unsatisfiable, hence  $p_a \Rightarrow p_b$  is valid.

#### Exercise 8.4 Exponential Behaviour

Show that the time complexity of our unification algorithm is exponential.

*Hint: construct a unification problem  $s_1 =? t_1, \dots, s_n =? t_n$ , such that the terms grow exponentially when applying the most general unifier.*

#### Solution

The exponential blowup can easily be seen with the following unification problem:

$$\begin{aligned}x_1 &=? f(x_2, x_2) \\x_2 &=? f(x_3, x_3) \\&\vdots \\x_n &=? f(x_{n+1}, x_{n+1}).\end{aligned}$$

#### Exercise 8.5 Unification

The code for unification in the book has some optimizations to improve its space usage. Develop a straightforward OCaml implementation of unification as it was presented in the lecture.