

# Logic

## Exercise Sheet 9

### Exercise 9.1 Resolution

Prove the following formula by resolution:

$$(\forall y. Q(f(a), f(y))) \wedge (\forall x y. Q(y, f(y)) \Rightarrow P(f(x), g(y, b))) \Rightarrow (\exists x. \exists y. \exists z. P(x, y) \wedge P(f(a), g(x, b)) \wedge Q(x, z))$$

### Solution

After skolemization and transformation into CNF, we get the following set of clauses, where variables are made unique:

$$\{Q(f(a), f(y_1))\} \tag{1}$$

$$\{\neg Q(y_2, f(y_2)), P(f(x_2), g(y_2, b))\} \tag{2}$$

$$\{\neg P(x_3, y_3), \neg P(f(a), g(x_3, b)), \neg Q(x_3, z_3)\} \tag{3}$$

Resolving (2) and (3) with the unifier  $\{x_2 \mapsto a, y_2 \mapsto x_3\}$  yields the resolvent

$$\{\neg Q(x_3, f(x_3)), \neg P(x_3, y_3), \neg Q(x_3, z_3)\} \tag{4}$$

Resolving (1) and (4) with the unifier  $\{x_3 \mapsto f(a), y_1 \mapsto f(a)\}$  yields the resolvent

$$\{\neg P(f(a), y_3), \neg Q(f(a), z_3)\} \tag{5}$$

Resolving (2) and (5) with the unifier  $\{x_2 \mapsto a, y_3 \mapsto g(y_2, b)\}$  yields the resolvent

$$\{\neg Q(y_2, f(y_2)), \neg Q(f(a), z_3)\} \tag{6}$$

Resolving (1) and (6) with the unifier  $\{y_2 \mapsto f(a), y_1 \mapsto f(a)\}$  yields the resolvent

$$\{\neg Q(f(a), z_3)\} \tag{7}$$

Finally, resolving (1) and (7) with the unifier  $\{z_3 \mapsto f(y_1)\}$  yields the empty clause.

### Exercise 9.2 Resolution clauses

For a set  $C$  of clauses, the set  $Res(C)$  is defined as the set of all clauses which can be derived from  $C$  by resolution. Find a finite clause set  $C$  such that  $Res(C)$  is infinite.

## Solution

One set of clauses is the following one:

$$\{P(c)\}, \{\neg P(x), P(f(x))\}$$

Resolution then produces the infinite list of resolvents  $P(f(c)), P(f(f(c))), \dots$

## Exercise 9.3 First-order Sequent Calculus

Use the sequent calculus of first-order logic to prove the following formulas:

- a)  $(\forall x.P \vee Q(x)) \Rightarrow (P \vee \forall x.Q(x))$
- b)  $\neg(\forall x.\exists y.\forall z.\neg P(x, z) \wedge P(z, y))$

## Solution

a)

$$\frac{\frac{\frac{\frac{\overline{(\forall x.P \vee Q(x)), P \vdash P, Q(x)}{Ax}}{\overline{(\forall x.P \vee Q(x)), P \vdash P, Q(x)}}{Ax}}{\overline{(\forall x.P \vee Q(x)), (P \vee Q(x)) \vdash P, Q(x)}}{\vee L}}{\overline{\forall x.P \vee Q(x) \vdash P, Q(x)}}{\vee L}}{\overline{\forall x.P \vee Q(x) \vdash P, \forall x.Q(x)}}{\vee R}}{\overline{\forall x.P \vee Q(x) \vdash P \vee \forall x.Q(x)}}{\vee R}}{\vdash (\forall x.P \vee Q(x)) \Rightarrow (P \vee \forall x.Q(x))}{\Rightarrow R}$$

b)

$$\frac{\frac{\frac{\frac{\frac{\overline{\Gamma, \Delta, \neg P(c, c), P(c, y), P(y, y) \vdash P(c, y)}{Ax}}{\overline{\Gamma, \Delta, \neg P(c, c), P(c, y), \neg P(c, y), P(y, y) \vdash \perp}}{\neg L}}{\overline{\Gamma, \Delta, \neg P(c, c), P(c, y), \neg P(c, y) \wedge P(y, y) \vdash \perp}}{\wedge L}}{\overline{\Gamma, \Delta, \neg P(c, c), P(c, y) \vdash \perp}}{\vee L (*)}}{\overline{\Gamma, \Delta, \neg P(c, c) \wedge P(c, y) \vdash \perp}}{\wedge L}}{\overline{\Gamma, \forall z.\neg P(c, z) \wedge P(z, y) \vdash \perp}}{\vee L}}{\overline{\Gamma, \exists y.\forall z.\neg P(c, z) \wedge P(z, y) \vdash \perp}}{\exists L}}{\overline{\forall x.\exists y.\forall z.\neg P(x, z) \wedge P(z, y) \vdash \perp}}{\vee L}}{\vdash \neg(\forall x.\exists y.\forall z.\neg P(x, z) \wedge P(z, y))}{\neg R}$$

where

$$\begin{aligned}\Gamma &= \forall x.\exists y.\forall z.\neg P(x, z) \wedge P(z, y) \\ \Delta &= \forall z.\neg P(c, z) \wedge P(z, y)\end{aligned}$$

(\*): This step works on the quantifier hidden in  $\Delta$ .

### Exercise 9.4 Resolution and Equality

- a) A monoid is an algebraic structure consisting of a set together with a binary, associative operation  $+$  and a distinguished neutral element  $0$ . These properties can be described by the following axioms:

$$\forall x. 0 + x = x$$

$$\forall x. x + 0 = x$$

$$\forall x. \forall y. \forall z. x + (y + z) = (x + y) + z$$

Prove by resolution that these axioms imply  $\forall x. 0 + (x + 0) = x$  in all normal models.

- b) A group is a monoid with inverse elements, i.e., additional to the monoid axioms, we have the following properties:

$$\forall x. \exists y. x + y = 0$$

$$\forall x. \exists y. y + x = 0$$

Prove by resolution that every element of a group has exactly one inverse.

### Solution

- a) An algebraic proof for this proposition is the chain of equalities  $0 + (a + 0) = a + 0 = a$  where the first and second axiom. We reuse this idea in the resolution proof.

The equality and monoid laws can be translated straightforwardly into clauses:

$$\{x = x\} \tag{1}$$

$$\{\neg(x = y), y = x\} \tag{2}$$

$$\{\neg(x = y), \neg(y = z), x = z\} \tag{3}$$

$$\{\neg(x_1 = y_1), \neg(x_2 = y_2), x_1 + x_2 = y_1 + y_2\} \tag{4}$$

$$\{0 + x = x\} \tag{5}$$

$$\{x + 0 = x\} \tag{6}$$

$$\{x + (y + z) = (x + y) + z\} \tag{7}$$

The conjecture corresponds to the clause  $\{0 + (a + 0) = a\}$ . Applying transitivity of equality (3) twice yields

$$\{\neg(0 + (a + 0) = x), \neg(x = y), \neg(y = a)\}$$

Application of (5) reduces this to

$$\{\neg(a + 0 = y), \neg(y = a)\}$$

Application of (6) then yields

$$\{\neg(a = a)\}$$

which, with (1), results in the empty clause.

- b) We want to show that  $\forall x.\forall y.\forall z.x + y = 0 \wedge x + z = 0 \Rightarrow y = z$ . The algebraic proof can be given as chain of equalities, for fixed  $a, b$  and  $c$ :

$$b = b + 0 = b + (a + c) = (b + a) + c = 0 + c = c$$

The resolution proof proceeds in a similar fashion as in the previous solution. Additionally to the group laws, we have two clauses representing the assumptions of the conjecture, i.e.  $\{a + b = 0\}$  and  $\{a + c = 0\}$ . The goal clause  $\{\neg(b = c)\}$  is expanded by transitivity of equality (3) into

$$\{\neg(b = w), \neg(w = x), \neg(x = y), \neg(y = z), \neg(z = c)\}$$

Applying symmetry (2) to (6) yields  $\{x = x + 0\}$ . Using this and (5) reduces the above clause to

$$\{\neg(b + 0 = x), \neg(x = y), \neg(y = 0 + c)\}$$

Application of the congruence rule (4) then yields

$$\{\neg(b = y_1), \neg(0 = y_2), \neg(y_1 + y_2 = y), \neg(y = 0 + c)\}$$

Reflexivity (1) gives

$$\{\neg(0 = y_2), \neg(b + y_2 = y), \neg(y = 0 + c)\}$$

With symmetry, we get  $\{0 = a + c\}$  from the second assumption. Then, we yield

$$\{\neg(b + (a + c) = y), \neg(y = 0 + c)\}$$

Using congruence (4) and the first assumption on the second formula reduces this clause to

$$\{\neg(b + (a + c) = (b + a) + c)\}$$

which yields the empty clause when resolving with (7).

## Proof Rules of the Sequent Calculus:

### Axioms

$$\frac{}{\Gamma, p \vdash \Delta, p} Ax \qquad \frac{}{\Gamma, \perp \vdash \Delta} \perp L \qquad \frac{}{\Gamma \vdash \Delta, \top} \top R$$

### Rules

$$\frac{\Gamma \vdash \Delta, p}{\Gamma, \neg p \vdash \Delta} \neg L$$

$$\frac{\Gamma, p \vdash \Delta}{\Gamma \vdash \Delta, \neg p} \neg R$$

$$\frac{\Gamma, p, q \vdash \Delta}{\Gamma, p \wedge q \vdash \Delta} \wedge L$$

$$\frac{\Gamma \vdash \Delta, p \quad \Gamma \vdash \Delta, q}{\Gamma \vdash \Delta, p \wedge q} \wedge R$$

$$\frac{\Gamma, p \vdash \Delta \quad \Gamma, q \vdash \Delta}{\Gamma, p \vee q \vdash \Delta} \vee L$$

$$\frac{\Gamma \vdash \Delta, p, q}{\Gamma \vdash \Delta, p \vee q} \vee R$$

$$\frac{\Gamma \vdash \Delta, p \quad \Gamma, q \vdash \Delta}{\Gamma, p \Rightarrow q \vdash \Delta} \Rightarrow L$$

$$\frac{\Gamma, p \vdash \Delta, q}{\Gamma \vdash \Delta, p \Rightarrow q} \Rightarrow R$$

$$\frac{\Gamma, \forall x.p, p[t/x] \vdash \Delta}{\Gamma, \forall x.p \vdash \Delta} \forall L$$

$$\frac{\Gamma \vdash \Delta, p[y/x]}{\Gamma \vdash \Delta, \forall x.p} \forall R$$

$y \notin FV(\Gamma \cup \Delta \cup (\forall x.p))$

$$\frac{\Gamma, p[y/x] \vdash \Delta}{\Gamma, \exists x.p \vdash \Delta} \exists L$$

$y \notin FV(\Gamma \cup \Delta \cup (\exists x.p))$

$$\frac{\Gamma \vdash \Delta, \exists x.p, p[t/x]}{\Gamma \vdash \Delta, \exists x.p} \exists R$$