Formalizing the Logic-Automaton Connection

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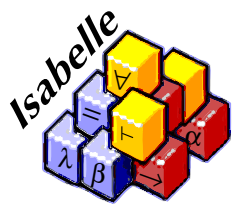

TPHOLs, 20.8.2009

Quiz

Which of these formulae are true (for natural numbers)?

$$\forall x{\geq}7.\ \exists y\ z.\ 3 * y + 5 * z = x$$

$$\forall x{\geq}8.\ \exists y\ z.\ 3 * y + 5 * z = x$$

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Stamp problem

Any postage of 8 cents or more can be made up using stamps of the denominations 3 cents and 5 cents.

Outline

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Quantifier elimination method

Decision Procedures for Presburger Arithmetic

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Algebraic e.g. Cooper's algorithm

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(also works for WS1S, see MONA)

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Reflection Write and verify the decision procedure as a recursive function in HOL itself (this talk).

Related Work

[Boutin, TACS 1997] Decision procedure for abelian rings in Coq (reflection)

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[Basin and Friedrich, FroCoS 2000] Combining WS1S and HOL (oracle)

Does Reflection work?

Hooking an 'oracle' to a theorem prover is risky business. The oracle could be buggy [...]. The only way to avoid a buggy oracle is to reconstruct a proof in the theorem prover based on output from the oracle, or perhaps verify the oracle itself. For a semantics based decision procedure, proof reconstruction is not a realistic option: one would have to formalize the entire automata-theoretic machinery within HOL [...].

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Our Claim

Verifying automata-based decision procedures in HOL is not as unrealistic as it may seem!

Outline

Presburger Arithmetic

Syntax (using de Bruijn indices)

datatype $pf = Eq\ (int\ list)\ int \mid Le\ (int\ list)\ int \mid And\ pf\ pf$
$\mid Or\ pf\ pf \mid Imp\ pf\ pf \mid Forall\ pf \mid Exist\ pf \mid Neg\ pf$

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Example: Stamp problem

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Encoding

$Forall\ (Imp\ (Le\ [-1]\ -8)\ (Exist\ (Exist\ (Eq\ [5,\ 3,\ -1]\ 0))))$

Evaluation

Diophantine (In)Equations

$$\textit{eval-dioph} :: \textit{int list} \Rightarrow \textit{nat list} \Rightarrow \textit{int}$$
$$\textit{eval-dioph}\ (k \cdot ks)\ (x \cdot xs) = k * \textit{int}\ x + \textit{eval-dioph}\ ks\ xs$$
$$\textit{eval-dioph}\ []\ xs = 0$$
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Formulae

$$\textit{eval-pf} :: \textit{pf} \Rightarrow \textit{nat list} \Rightarrow \textit{bool}$$
$$\textit{eval-pf}\ (\textit{Eq}\ ks\ l)\ xs = (\textit{eval-dioph}\ ks\ xs = l)$$
$$\textit{eval-pf}\ (\textit{Le}\ ks\ l)\ xs = (\textit{eval-dioph}\ ks\ xs \leq l)$$
$$\textit{eval-pf}\ (\textit{Neg}\ p)\ xs = (\neg\ \textit{eval-pf}\ p\ xs)$$
$$\textit{eval-pf}\ (\textit{And}\ p\ q)\ xs = (\textit{eval-pf}\ p\ xs \wedge \textit{eval-pf}\ q\ xs)$$
$$\textit{eval-pf}\ (\textit{Or}\ p\ q)\ xs = (\textit{eval-pf}\ p\ xs \vee \textit{eval-pf}\ q\ xs)$$
$$\textit{eval-pf}\ (\textit{Forall}\ p)\ xs = (\forall x.\ \textit{eval-pf}\ p\ (x \cdot xs))$$
$$\textit{eval-pf}\ (\textit{Exist}\ p)\ xs = (\exists x.\ \textit{eval-pf}\ p\ (x \cdot xs))$$

Automata on Bit Vectors

Input symbols of an automaton corresponding to a formula with n free variables $x_0, \ldots, x_{n-1}$ are bit lists of length n:

$$\begin{matrix} x_0 \\ \vdots \\ x_i \\ \vdots \\ x_{n-1} \end{matrix} \left[\left[\begin{matrix} b_{0,0} \\ \vdots \\ b_{i,0} \\ \vdots \\ b_{n-1,0} \end{matrix}\right] \cdots \left[\begin{matrix} b_{0,j} \\ \vdots \\ b_{i,j} \\ \vdots \\ b_{n-1,j} \end{matrix}\right] \cdots \left[\begin{matrix} b_{0,m-1} \\ \vdots \\ b_{i,m-1} \\ \vdots \\ b_{n-1,m-1} \end{matrix}\right]\right]$$

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[Boudet and Comon, CAAP 1996]

Values of Variables

List of variables (encoded as list of column vectors)

$$
\begin{aligned}
&\mathit{nats\text{-}of\text{-}boolss} :: \mathit{nat} \Rightarrow \mathit{bool\ list\ list} \Rightarrow \mathit{nat\ list} \\
&\mathit{nats\text{-}of\text{-}boolss}\ n\ [] = \mathit{replicate}\ n\ 0 \\
&\mathit{nats\text{-}of\text{-}boolss}\ n\ (bs \cdot bss) = \\
&\quad \mathit{map}\ (\lambda(b, x).\ \mathit{nat\text{-}of\text{-}bool}\ b + 2 * x) \\
&\quad\ (\mathit{zip}\ bs\ (\mathit{nats\text{-}of\text{-}boolss}\ n\ bss))
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Single variable (encoded as row vector)

$$
\begin{aligned}
&\mathit{nat\text{-}of\text{-}bools} :: \mathit{bool\ list} \Rightarrow \mathit{nat} \\
&\mathit{nat\text{-}of\text{-}bools}\ [] = 0 \\
&\mathit{nat\text{-}of\text{-}bools}\ (b \cdot bs) = \mathit{nat\text{-}of\text{-}bool}\ b + 2 * \mathit{nat\text{-}of\text{-}bools}\ bs
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Deterministic Automata

Represented by type

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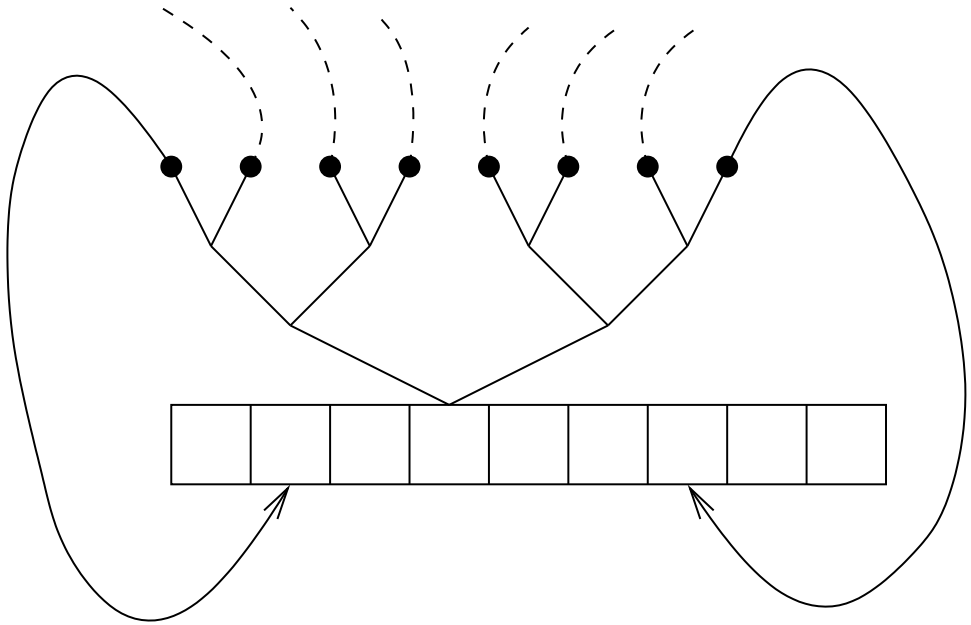
Transition table



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Note: (finite) sets of states represented as bitstrings

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Better: only generate reachable states (using DFS)

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Existential quantifiers

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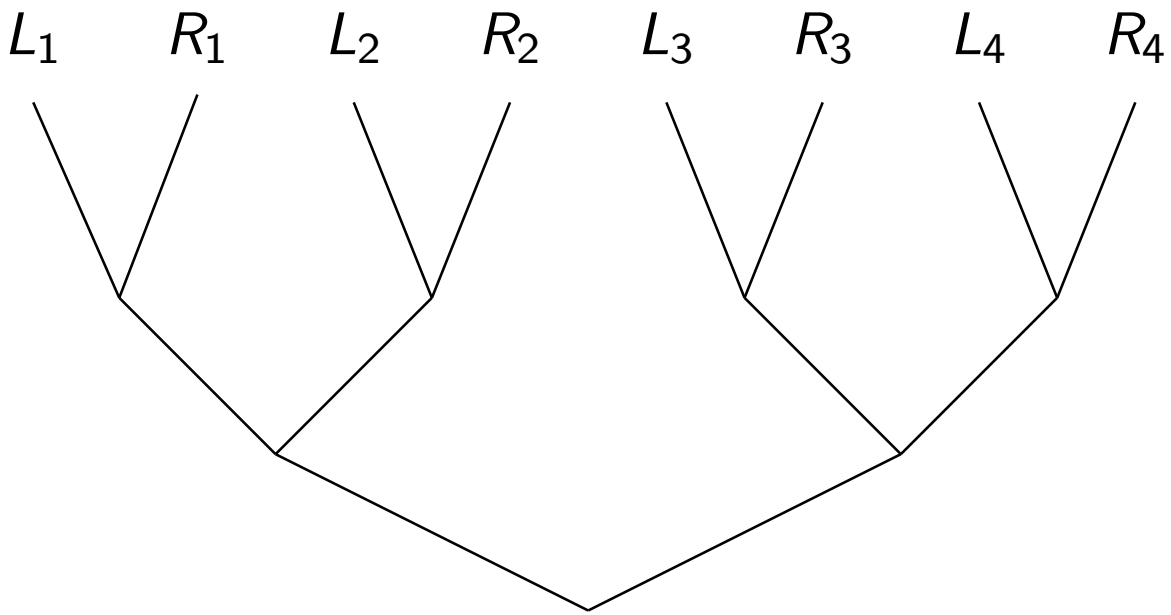
Universal quantifiers

Note: $(\forall\, x.\ P\ x) = (\neg\ (\exists\, x.\ \neg\ P\ x))$

Projection



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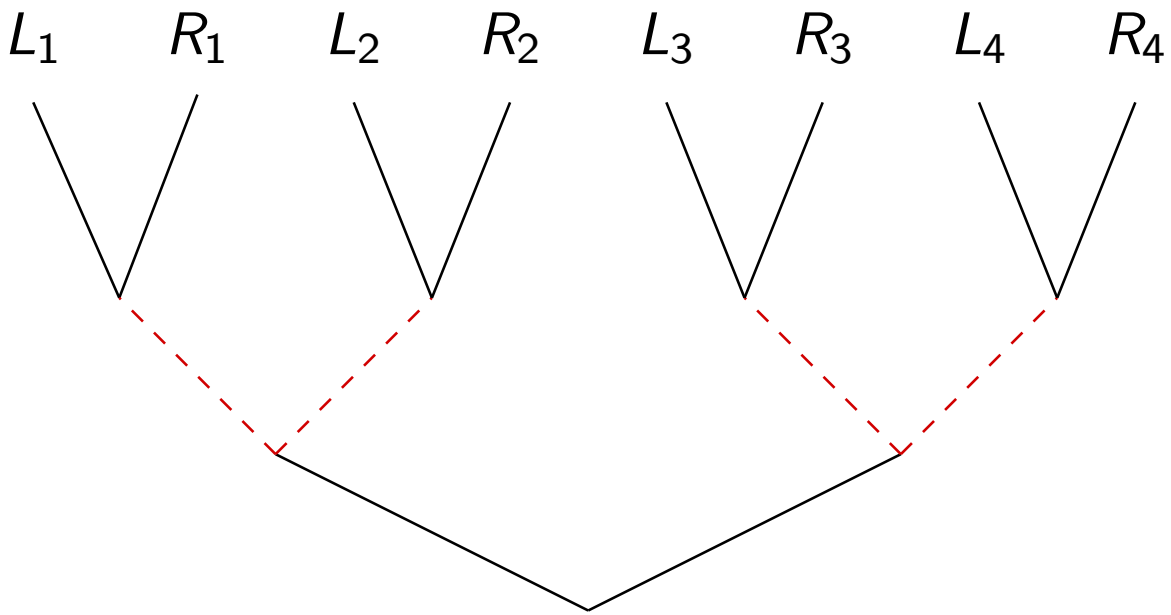

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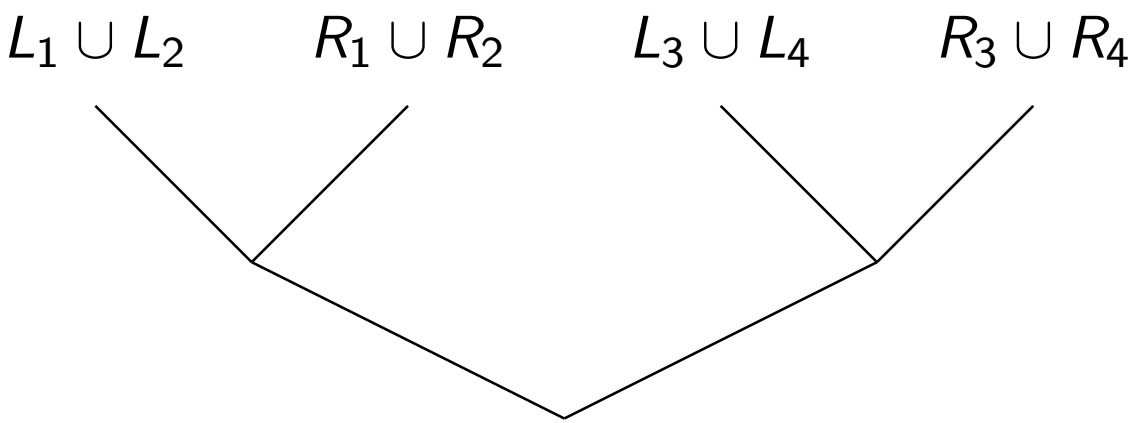

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Diophantine Equations

Method by Boudet and Comon

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&(\mathit{eval\text{-}dioph}\ \mathit{ks}\ (\mathit{map}\ (\lambda x.\ x\ \mathit{mod}\ 2)\ \mathit{xs})\ \mathit{mod}\ 2 = \mathit{l}\ \mathit{mod}\ 2 \wedge \\
&\ \mathit{eval\text{-}dioph}\ \mathit{ks}\ (\mathit{map}\ (\lambda x.\ x\ \mathit{div}\ 2)\ \mathit{xs}) = \\
&\ (\mathit{l} - \mathit{eval\text{-}dioph}\ \mathit{ks}\ (\mathit{map}\ (\lambda x.\ x\ \mathit{mod}\ 2)\ \mathit{xs}))\ \mathit{div}\ 2)
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xs is a solution iff...

- ...it is a solution modulo *2*, and...

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- ...quotient of *xs* and *2* is a solution of another equation with same coefficients, but different right-hand side.

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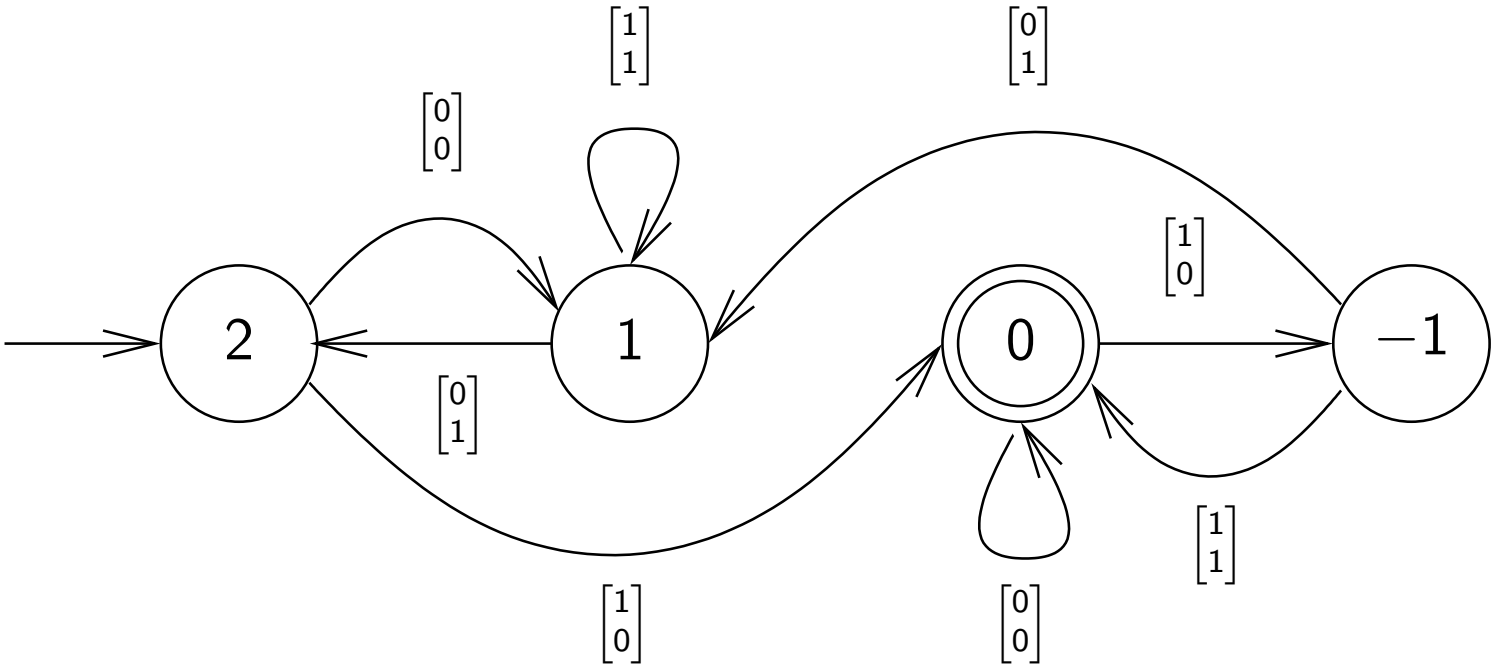
Reachable right-hand sides are bounded

$\mathit{If}\ |m| \leq \mathit{max}\ |l|\ (\sum k{\leftarrow}\mathit{ks}.\ |k|)\ \ \mathit{then}$
$|(m - \mathit{eval\text{-}dioph}\ \mathit{ks}\ (\mathit{map}\ (\lambda x.\ x\ \mathit{mod}\ 2)\ \mathit{xs}))\ \mathit{div}\ 2|$
$\leq \mathit{max}\ |l|\ (\sum k{\leftarrow}\mathit{ks}.\ |k|).$

Diophantine Equations — Example

Formula: $2x - 3y = 2$



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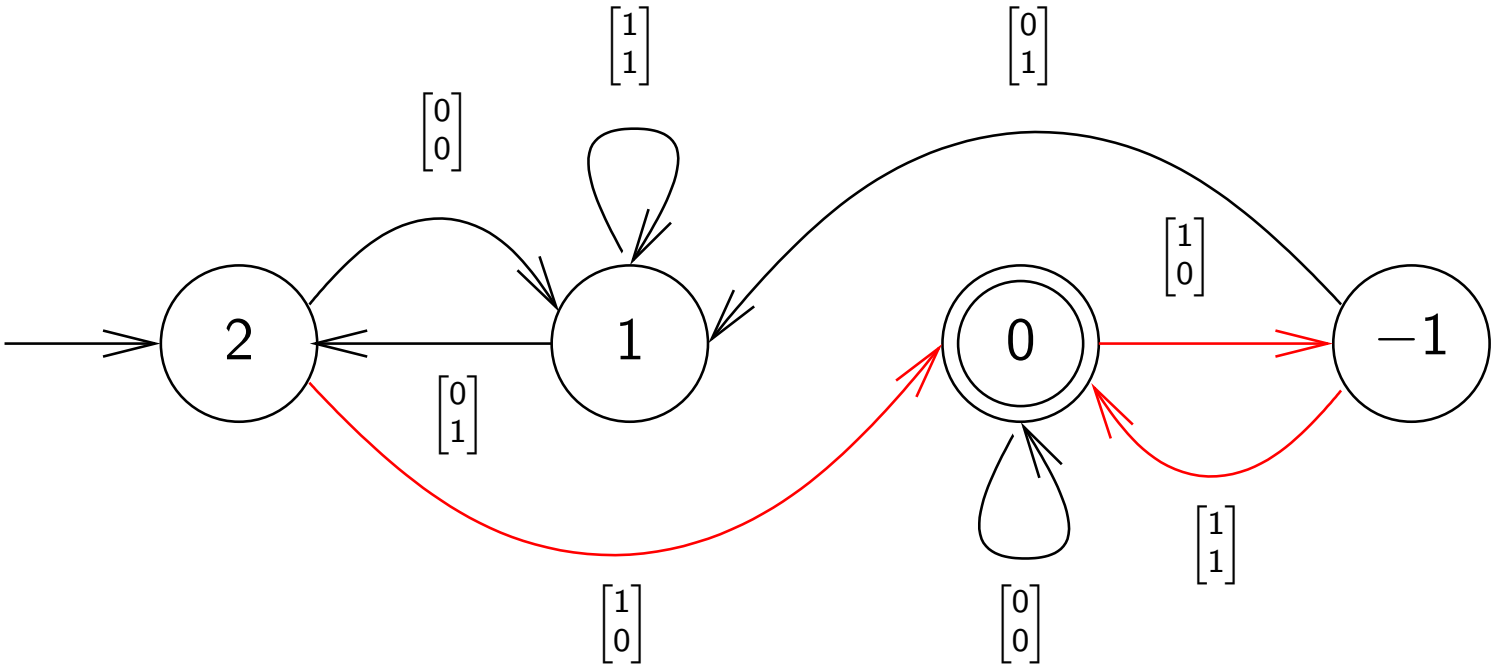

Some solutions

1	2	4	8	16	32	64	128	
1	1	1						$7 \cdot 2 = 14$
0	0	1						$4 \cdot 3 = 12$

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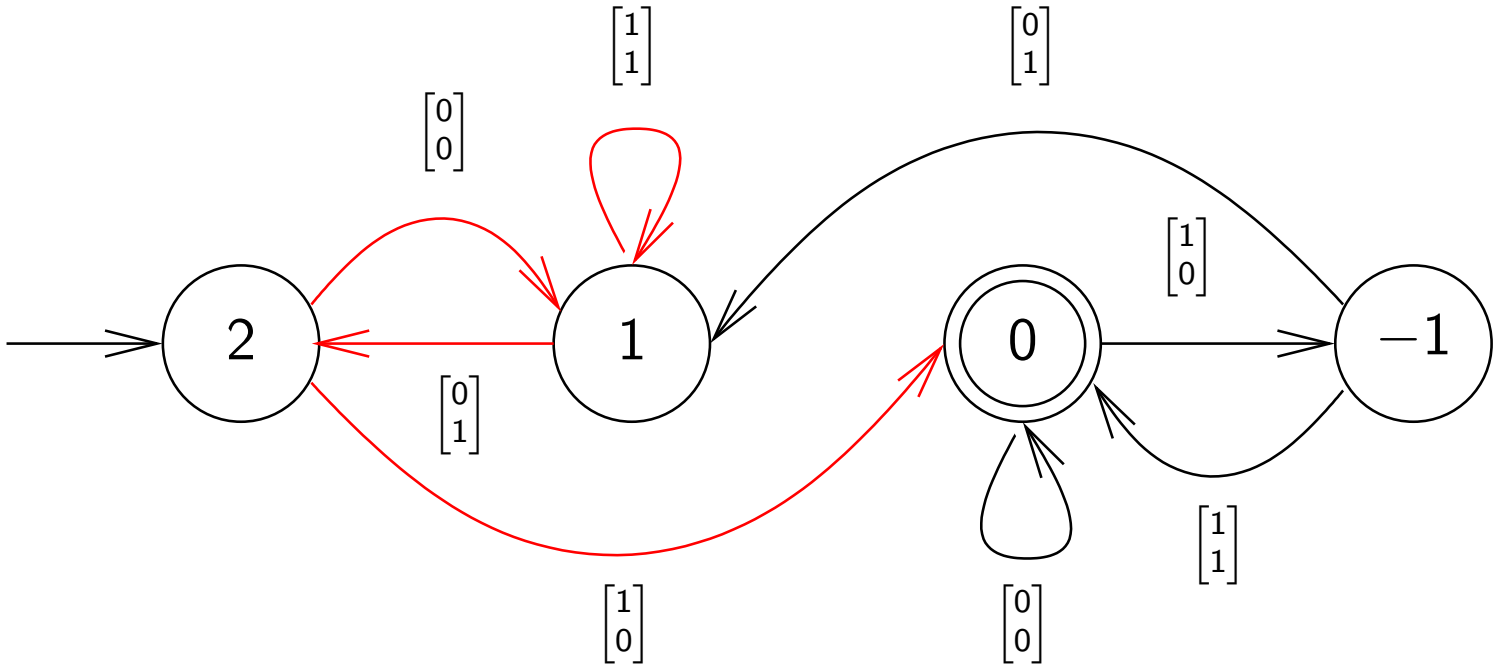
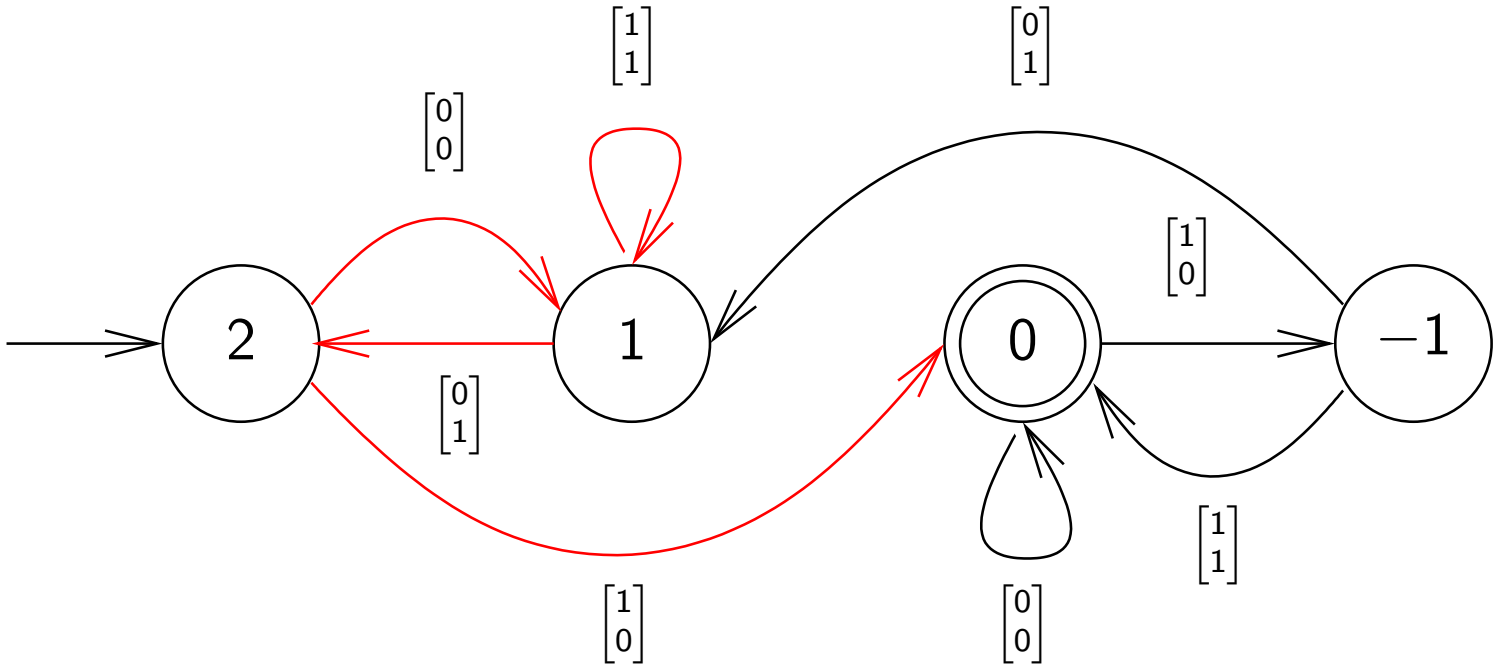


Some solutions

1	2	4	8	16	32	64	128	
0	1	1	0	1				$22 \cdot 2 = 44$
0	1	1	1	0				$14 \cdot 3 = 42$

Diophantine Equations — Example

Formula: $2x - 3y = 2$



Some solutions

1	2	4	8	16	32	64	128	
0	1	0	1					$10 \cdot 2 = 20$
0	1	1	0					$6 \cdot 3 = 18$

Diophantine Equations — Example

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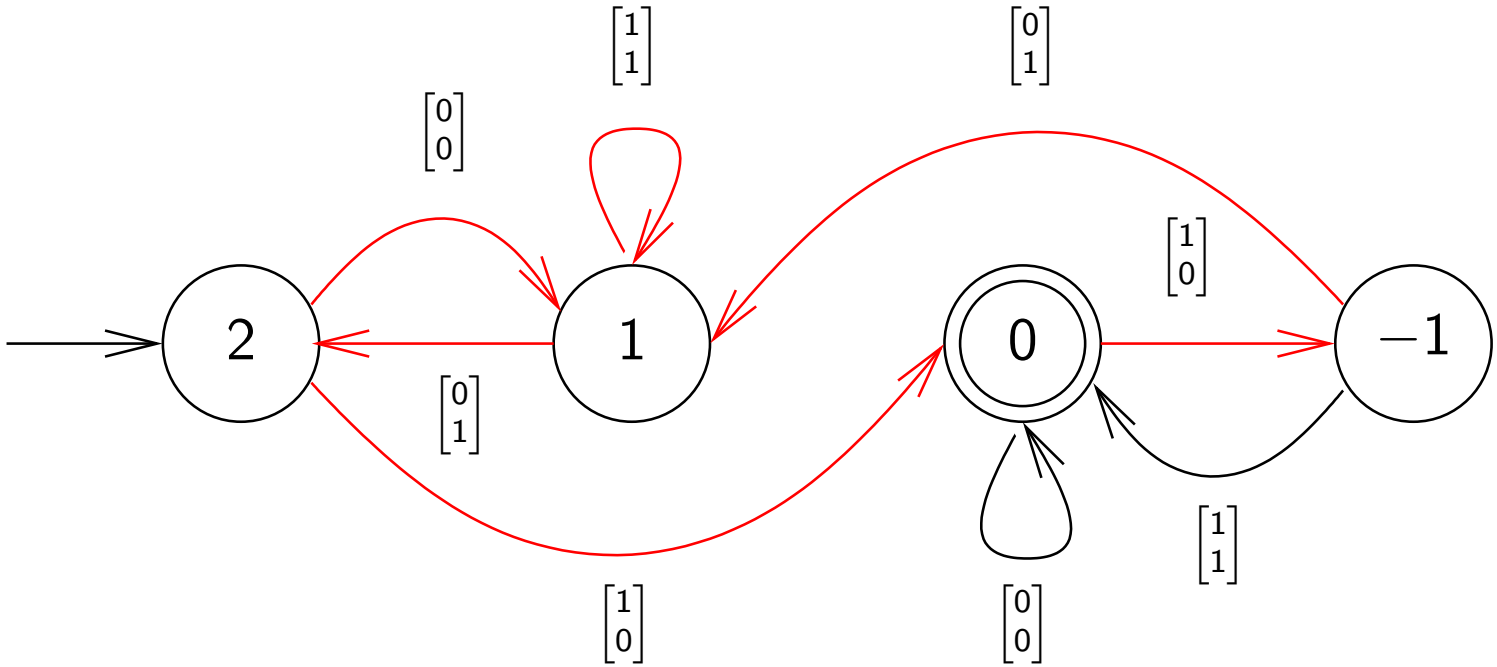


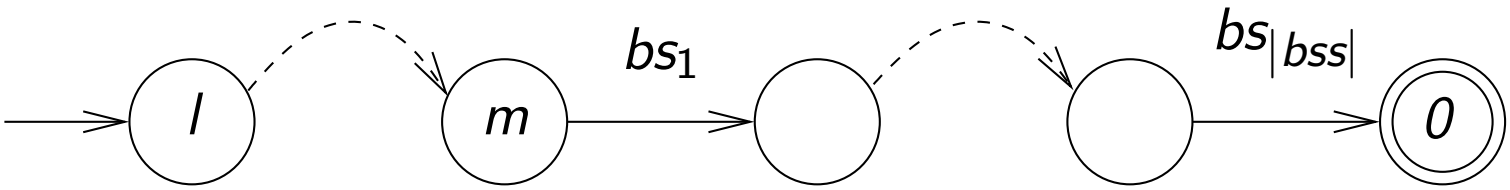
Some solutions

1	2	4	8	16	32	64	128	
0	1	0	1	1	0	0	1	$154 \cdot 2 = 308$
0	1	1	0	0	1	1	0	$102 \cdot 3 = 306$

Diophantine Equations — The Code

```
eq-dfa :: nat ⇒ int list ⇒ int ⇒ dfa
eq-dfa n ks l ≡
let (is, js) = dioph-dfs n ks l
in (map (λj. make-bdd
                (λxs. if eval-dioph ks xs mod 2 = j mod 2
                      then the is[int-to-nat-bij
                                    ((j − eval-dioph ks xs) div 2)]
                      else |js|)
                n [])
    js @
    [Leaf |js|],
    map (λj. j = 0) js @ [False])
```

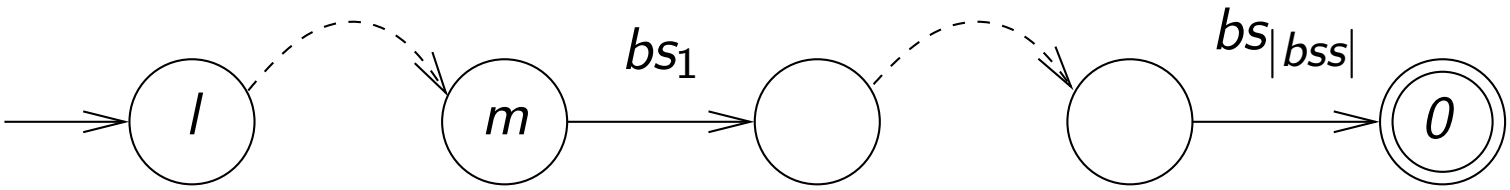
Diophantine Equations — Correctness



Strengthened statement

If $(l, m) \in (succsr\ (dioph\text{-}succs\ n\ ks))^*$ *and*
$\forall\, bs{\in}bss.\ is\text{-}alph\ n\ bs$ *then*
dfa-accepting $(eq\text{-}dfa\ n\ ks\ l)$
$(dfa\text{-}steps\ (eq\text{-}dfa\ n\ ks\ l)$
$(the\ (fst\ (dioph\text{-}dfs\ n\ ks\ l))[int\text{-}to\text{-}nat\text{-}bij\ m])\ bss) =$
$(eval\text{-}dioph\ ks\ (nats\text{-}of\text{-}boolss\ n\ bss) = m)$.

Diophantine Equations — Correctness



Corollary ($l = m$)

If $\forall$ *bs*$\in$*bss. is-alph n bs then*
dfa-accepts (*eq-dfa n ks l*) *bss* =
(*eval-dioph ks* (*nats-of-boolss n bss*) = *l*).

Outline

The Decision Procedure

```
dfa-of-pf :: nat ⇒ pf ⇒ dfa
dfa-of-pf n (Eq ks l) = eq-dfa n ks l
dfa-of-pf n (Le ks l) = ineq-dfa n ks l
dfa-of-pf n (Neg p) = negate-dfa (dfa-of-pf n p)
dfa-of-pf n (And p q) =
  binop-dfa (∧) (dfa-of-pf n p) (dfa-of-pf n q)
dfa-of-pf n (Or p q) =
  binop-dfa (∨) (dfa-of-pf n p) (dfa-of-pf n q)
dfa-of-pf n (Exist p) =
  rquot (det-nfa (quantify-nfa 0 (nfa-of-dfa (dfa-of-pf (Suc n) p))))
   n
dfa-of-pf n (Forall p) = dfa-of-pf n (Neg (Exist (Neg p)))
```

The Decision Procedure

$$
\begin{aligned}
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&\textit{dfa-of-pf}\ n\ (\textit{Eq}\ \textit{ks}\ l) = \textit{eq-dfa}\ n\ \textit{ks}\ l\\
&\textit{dfa-of-pf}\ n\ (\textit{Le}\ \textit{ks}\ l) = \textit{ineq-dfa}\ n\ \textit{ks}\ l\\
&\textit{dfa-of-pf}\ n\ (\textit{Neg}\ p) = \textit{negate-dfa}\ (\textit{dfa-of-pf}\ n\ p)\\
&\textit{dfa-of-pf}\ n\ (\textit{And}\ p\ q) =\\
&\quad \textit{binop-dfa}\ (\wedge)\ (\textit{dfa-of-pf}\ n\ p)\ (\textit{dfa-of-pf}\ n\ q)\\
&\textit{dfa-of-pf}\ n\ (\textit{Or}\ p\ q) =\\
&\quad \textit{binop-dfa}\ (\vee)\ (\textit{dfa-of-pf}\ n\ p)\ (\textit{dfa-of-pf}\ n\ q)\\
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&\quad\ n\\
&\textit{dfa-of-pf}\ n\ (\textit{Forall}\ p) = \textit{dfa-of-pf}\ n\ (\textit{Neg}\ (\textit{Exist}\ (\textit{Neg}\ p)))
\end{aligned}
$$

Theorem (Correctness)

$$
\begin{aligned}
&\textit{If}\ \forall\, \textit{bs} \in \textit{bss}.\ \textit{is-alph}\ n\ \textit{bs}\ \ \textit{then}\\
&\textit{dfa-accepts}\ (\textit{dfa-of-pf}\ n\ p)\ \textit{bss} = \textit{eval-pf}\ p\ (\textit{nats-of-boolss}\ n\ \textit{bss}).
\end{aligned}
$$

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<tr><td rowspan="2">Forall
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- Use of DFS pays off!

Future Work

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- Extend to WS1S, and apply it to circuit verification problems described in [Basin and Friedrich, FROCOS 2000].