Formalizing the Logic-Automaton Connection

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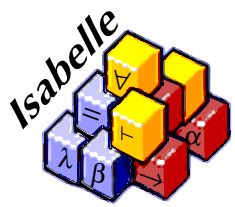
Institut für Informatik
Technische Universität München



Castro Urdiales, 22.4.2010

Quiz

Which of these formulae are true (for natural numbers)?

$$\forall x{\geq}7.\ \exists y\ z.\ 3 * y + 5 * z = x$$

$$\forall x{\geq}8.\ \exists y\ z.\ 3 * y + 5 * z = x$$

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Stamp problem

Any postage of 8 cents or more can be made up using stamps of the denominations 3 cents and 5 cents.

Outline

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Quantifier elimination method

Decision Procedures for Presburger Arithmetic

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Algebraic e.g. Cooper's algorithm

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Semantic e.g. using automata on bitstrings (this talk)
(also works for WS1S, see MONA)

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Reflection Write and verify the decision procedure as a recursive function in HOL itself (this talk).

Related Work

[Boutin, TACS 1997] Decision procedure for abelian rings in Coq (reflection)

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[Basin and Friedrich, FroCoS 2000] Combining WS1S and HOL (oracle)

Does Reflection work?

Hooking an 'oracle' to a theorem prover is risky business. The oracle could be buggy [...]. The only way to avoid a buggy oracle is to reconstruct a proof in the theorem prover based on output from the oracle, or perhaps verify the oracle itself. For a semantics based decision procedure, proof reconstruction is not a realistic option: one would have to formalize the entire automata-theoretic machinery within HOL [...].

[Basin and Friedrich, FroCoS 2000]

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Our Claim

Verifying automata-based decision procedures in HOL is not as unrealistic as it may seem!

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Reflection in a Nutshell

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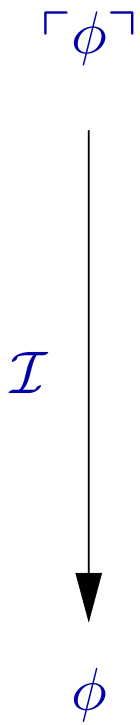

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$\mathcal{I} : R \Rightarrow bool$	Interpretation

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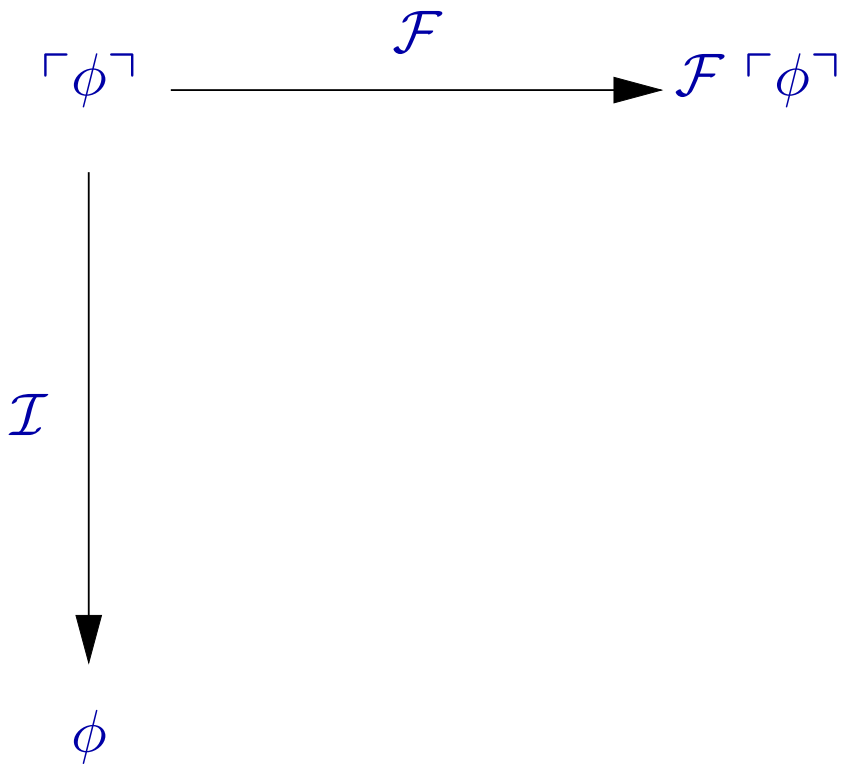

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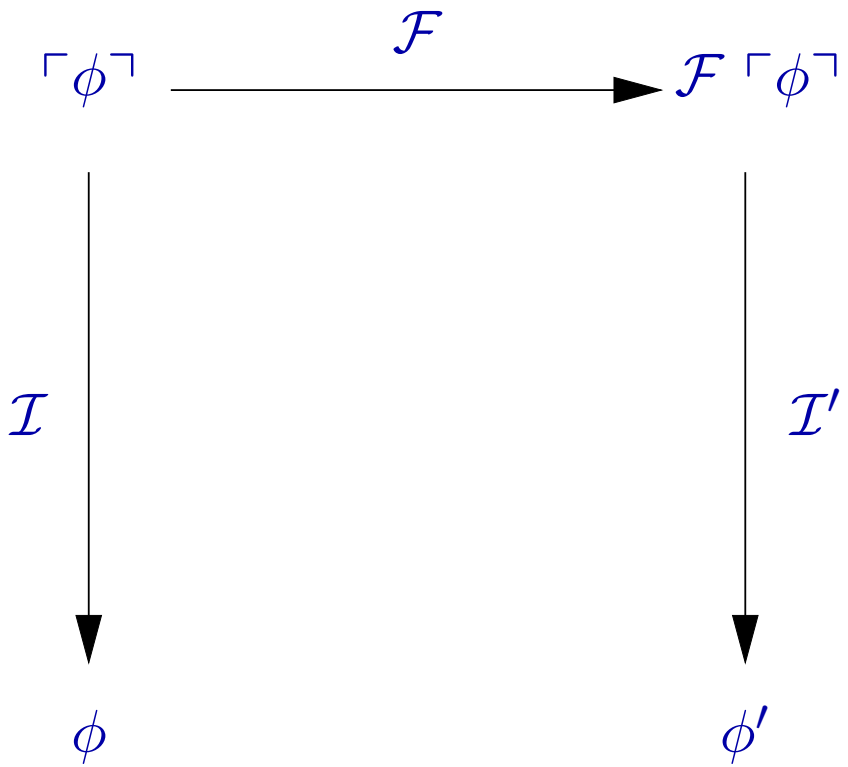

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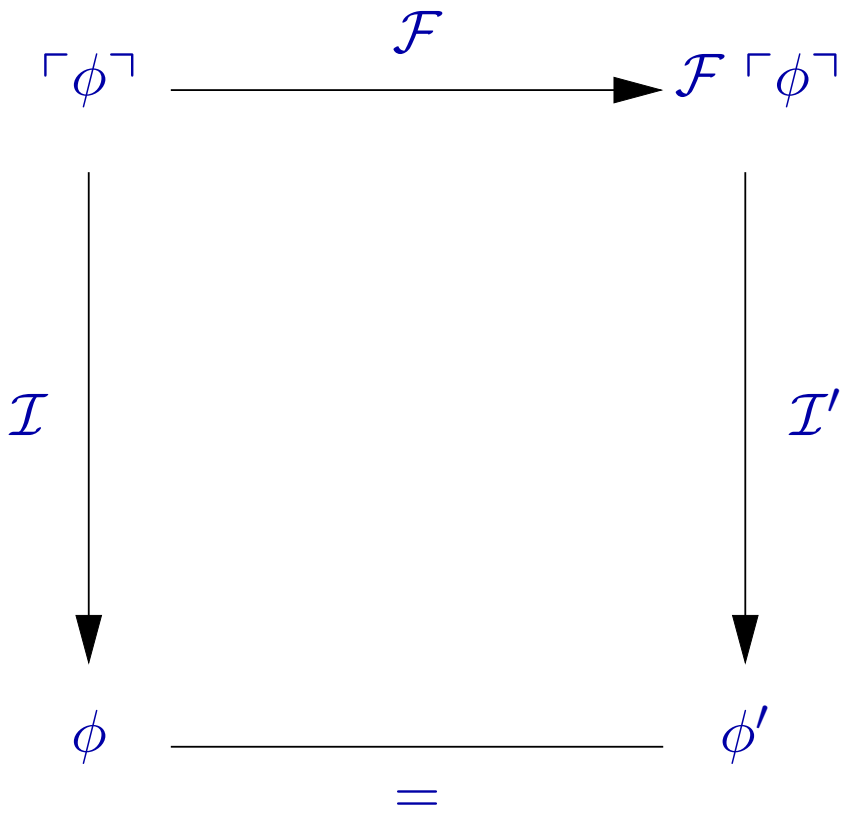

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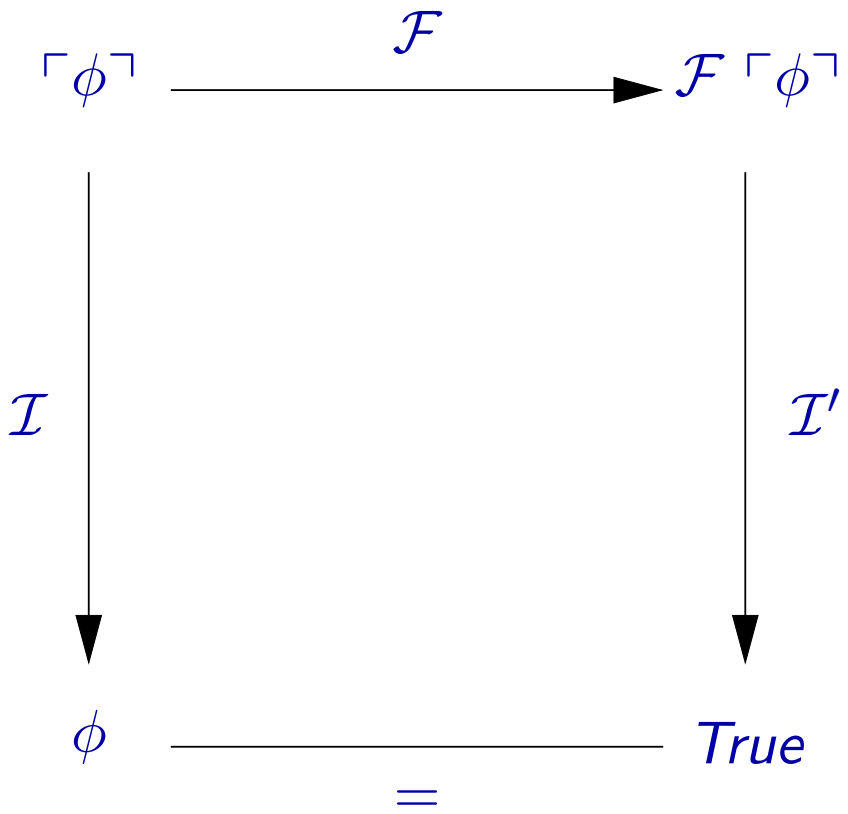

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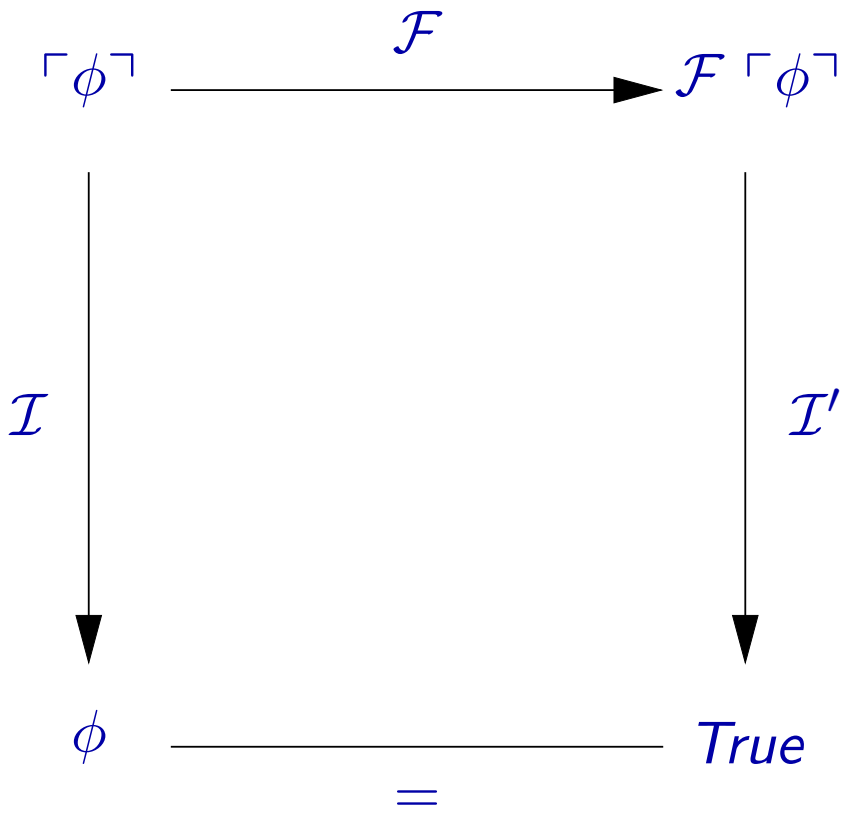

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$$\mathcal{I}\ r = \mathcal{I}'\ (\mathcal{F}\ r)$$

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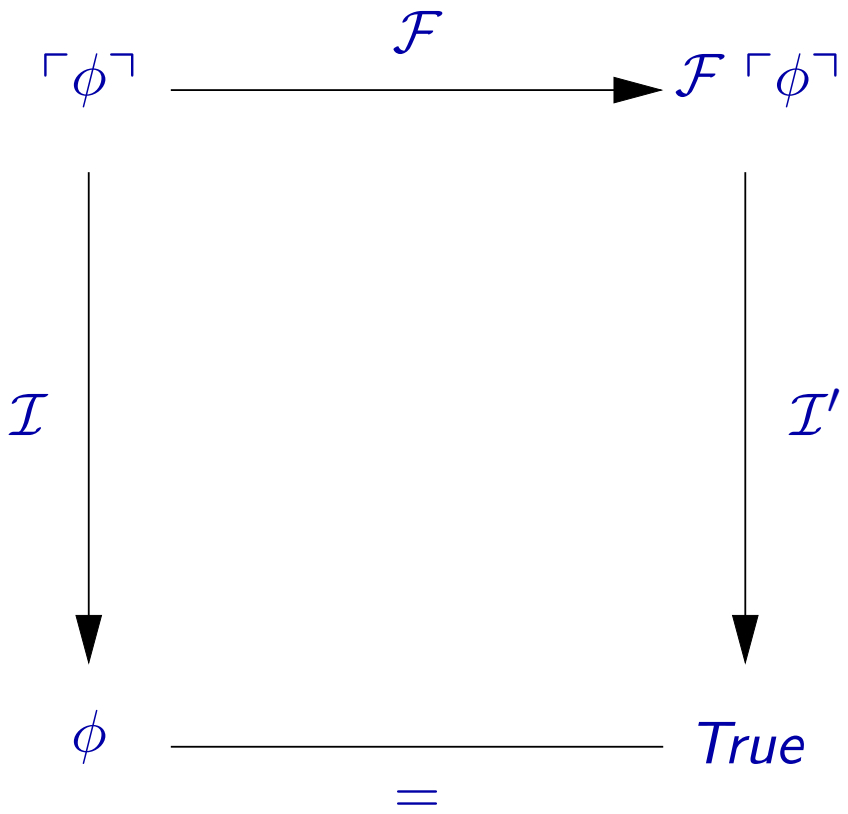

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[Geuvers et al., TPHOLs 2000]

Presburger Arithmetic

Syntax (using de Bruijn indices)

datatype $pf = Eq\ (int\ list)\ int \mid Le\ (int\ list)\ int \mid And\ pf\ pf$
$\mid Or\ pf\ pf \mid Imp\ pf\ pf \mid Forall\ pf \mid Exist\ pf \mid Neg\ pf$

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Example: Stamp problem

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Encoding

$Forall\ (Imp\ (Le\ [-1]\ -8)\ (Exist\ (Exist\ (Eq\ [5, 3, -1]\ 0))))$

Evaluation

Diophantine (In)Equations

$$\textit{eval-dioph} :: \textit{int list} \Rightarrow \textit{nat list} \Rightarrow \textit{int}$$
$$\textit{eval-dioph}\ (k \cdot ks)\ (x \cdot xs) = k * \textit{int}\ x + \textit{eval-dioph}\ ks\ xs$$
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Formulae

$$\textit{eval-pf} :: \textit{pf} \Rightarrow \textit{nat list} \Rightarrow \textit{bool}$$
$$\textit{eval-pf}\ (\textit{Eq}\ ks\ l)\ xs = (\textit{eval-dioph}\ ks\ xs = l)$$
$$\textit{eval-pf}\ (\textit{Le}\ ks\ l)\ xs = (\textit{eval-dioph}\ ks\ xs \leq l)$$
$$\textit{eval-pf}\ (\textit{Neg}\ p)\ xs = (\neg\ \textit{eval-pf}\ p\ xs)$$
$$\textit{eval-pf}\ (\textit{And}\ p\ q)\ xs = (\textit{eval-pf}\ p\ xs \wedge \textit{eval-pf}\ q\ xs)$$
$$\textit{eval-pf}\ (\textit{Or}\ p\ q)\ xs = (\textit{eval-pf}\ p\ xs \vee \textit{eval-pf}\ q\ xs)$$
$$\textit{eval-pf}\ (\textit{Forall}\ p)\ xs = (\forall x.\ \textit{eval-pf}\ p\ (x \cdot xs))$$
$$\textit{eval-pf}\ (\textit{Exist}\ p)\ xs = (\exists x.\ \textit{eval-pf}\ p\ (x \cdot xs))$$

Automata on Bit Vectors

Input symbols of an automaton corresponding to a formula with n free variables $x_0, \ldots, x_{n-1}$ are bit lists of length n:

$$\begin{matrix} x_0 \\ \vdots \\ x_i \\ \vdots \\ x_{n-1} \end{matrix} \left[\left[\begin{matrix} b_{0,0} \\ \vdots \\ b_{i,0} \\ \vdots \\ b_{n-1,0} \end{matrix}\right] \cdots \left[\begin{matrix} b_{0,j} \\ \vdots \\ b_{i,j} \\ \vdots \\ b_{n-1,j} \end{matrix}\right] \cdots \left[\begin{matrix} b_{0,m-1} \\ \vdots \\ b_{i,m-1} \\ \vdots \\ b_{n-1,m-1} \end{matrix}\right]\right]$$

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[Boudet and Comon, CAAP 1996]

Values of Variables

List of variables (encoded as list of column vectors)

$$
\begin{aligned}
&\textit{nats-of-boolss} :: \textit{nat} \Rightarrow \textit{bool list list} \Rightarrow \textit{nat list} \\
&\textit{nats-of-boolss}\ n\ [] = \textit{replicate}\ n\ 0 \\
&\textit{nats-of-boolss}\ n\ (bs \cdot bss) = \\
&\quad \textit{map}\ (\lambda(b, x).\ \textit{nat-of-bool}\ b + 2 * x) \\
&\quad (\textit{zip}\ bs\ (\textit{nats-of-boolss}\ n\ bss))
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Single variable (encoded as row vector)

$$
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\end{aligned}
$$

Represented by type

$$dfa = \underbrace{nat\ bdd\ list}_{\text{transition table}} \times \underbrace{bool\ list}_{\text{accepting states}}$$

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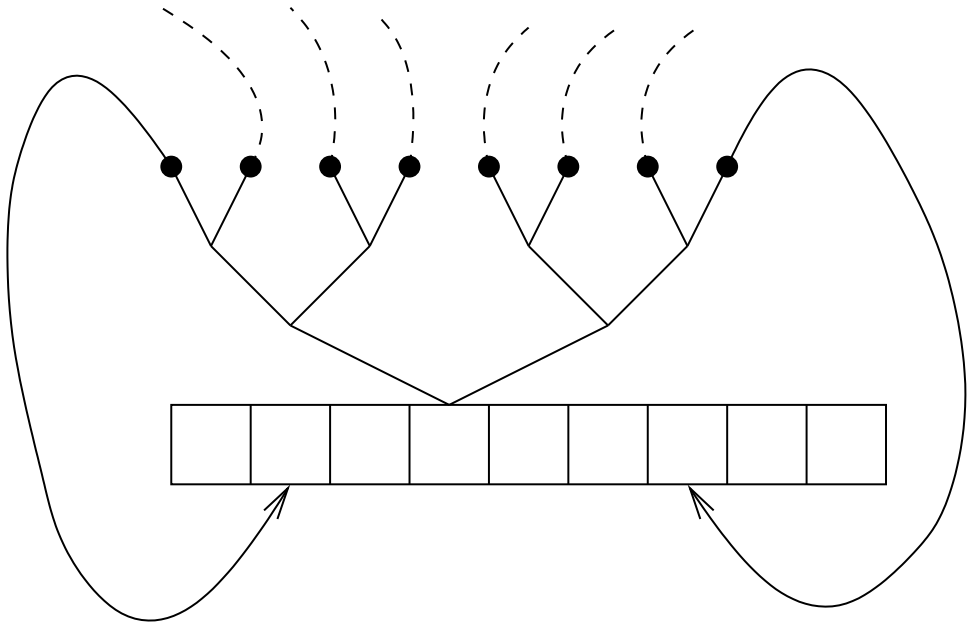
Transition table



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Note: (finite) sets of states represented as bitstrings

BDDs

Purely functional BDDs (no sharing!)

datatype α *bdd* $=$ *Leaf* α | *Branch* (α *bdd*) (α *bdd*)

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Lookup

bdd-lookup :: α *bdd* $\Rightarrow$ *bool list* $\Rightarrow$ α
bdd-lookup (*Leaf x*) *bs* = *x*
bdd-lookup (*Branch l r*) (*b* · *bs*) =
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Applying binary operators

bdd-binop :: ($\alpha \Rightarrow \beta \Rightarrow \gamma$) $\Rightarrow$ α *bdd* $\Rightarrow$ β *bdd* $\Rightarrow$ γ *bdd*
If bddh |*bs*| *l and bddh* |*bs*| *r then*
bdd-lookup (*bdd-binop f l r*) *bs* =
  f (*bdd-lookup l bs*) (*bdd-lookup r bs*).

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Better: only generate reachable states (using DFS)

Depth First Search

Generic DFS function [Nishihara and Minamide, AFP 2004]

$dfs :: \beta \Rightarrow \alpha\ list \Rightarrow \beta$ (α: node, β: visited nodes)

$dfs\ S\ [] = S$

$dfs\ S\ (x \cdot xs) =$

$(if\ memb\ x\ S\ then\ dfs\ S\ xs\ else\ dfs\ (ins\ x\ S)\ (succs\ x\ @\ xs))$

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Parameters for DFS

$succs :: \alpha \Rightarrow \alpha\ list$

$empt :: \beta$

$is\text{-}node :: \alpha \Rightarrow bool$

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$invariant :: \beta \Rightarrow bool$

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$invariant :: \beta \Rightarrow bool$

Correctness

If $is\text{-}node\ y$ *and* $is\text{-}node\ x$ *then*

$memb\ y\ (dfs\ empt\ [x]) = ((x, y) \in succs^*)$.

Diophantine Equations

Method by Boudet and Comon

$$
\begin{aligned}
&(\mathit{eval\text{-}dioph}\ ks\ xs = l) = \\
&(\mathit{eval\text{-}dioph}\ ks\ (\mathit{map}\ (\lambda x.\ x\ \mathit{mod}\ 2)\ xs)\ \mathit{mod}\ 2 = l\ \mathit{mod}\ 2 \wedge \\
&\ \mathit{eval\text{-}dioph}\ ks\ (\mathit{map}\ (\lambda x.\ x\ \mathit{div}\ 2)\ xs) = \\
&\ (l - \mathit{eval\text{-}dioph}\ ks\ (\mathit{map}\ (\lambda x.\ x\ \mathit{mod}\ 2)\ xs))\ \mathit{div}\ 2)
\end{aligned}
$$

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Method by Boudet and Comon

$$(\mathit{eval\text{-}dioph}\ \mathit{ks}\ \mathit{xs} = \mathit{l}) =$$
$$(\mathit{eval\text{-}dioph}\ \mathit{ks}\ (\mathit{map}\ (\lambda x.\ x\ \mathit{mod}\ 2)\ \mathit{xs})\ \mathit{mod}\ 2 = \mathit{l}\ \mathit{mod}\ 2 \wedge$$
$$\mathit{eval\text{-}dioph}\ \mathit{ks}\ (\mathit{map}\ (\lambda x.\ x\ \mathit{div}\ 2)\ \mathit{xs}) =$$
$$(\mathit{l} - \mathit{eval\text{-}dioph}\ \mathit{ks}\ (\mathit{map}\ (\lambda x.\ x\ \mathit{mod}\ 2)\ \mathit{xs}))\ \mathit{div}\ 2)$$

xs is a solution iff...

- ...it is a solution modulo *2*, and...

Diophantine Equations

Method by Boudet and Comon

$$
\begin{aligned}
&(\mathit{eval\text{-}dioph}\ \mathit{ks}\ \mathit{xs} = l) = \\
&(\mathit{eval\text{-}dioph}\ \mathit{ks}\ (\mathit{map}\ (\lambda x.\ x\ \mathit{mod}\ 2)\ \mathit{xs})\ \mathit{mod}\ 2 = l\ \mathit{mod}\ 2 \wedge \\
&\ \mathit{eval\text{-}dioph}\ \mathit{ks}\ (\mathit{map}\ (\lambda x.\ x\ \mathit{div}\ 2)\ \mathit{xs}) = \\
&\ (l - \mathit{eval\text{-}dioph}\ \mathit{ks}\ (\mathit{map}\ (\lambda x.\ x\ \mathit{mod}\ 2)\ \mathit{xs}))\ \mathit{div}\ 2)
\end{aligned}
$$

xs is a solution iff...

- ...it is a solution modulo *2*, and...
- ...quotient of *xs* and *2* is a solution of another equation with same coefficients, but different right-hand side.

Diophantine Equations

Method by Boudet and Comon

$$(\mathit{eval\text{-}dioph}\ ks\ xs = l) =$$
$$(\mathit{eval\text{-}dioph}\ ks\ (\mathit{map}\ (\lambda x.\ x\ \mathit{mod}\ 2)\ xs)\ \mathit{mod}\ 2 = l\ \mathit{mod}\ 2 \wedge$$
$$\mathit{eval\text{-}dioph}\ ks\ (\mathit{map}\ (\lambda x.\ x\ \mathit{div}\ 2)\ xs) =$$
$$(l - \mathit{eval\text{-}dioph}\ ks\ (\mathit{map}\ (\lambda x.\ x\ \mathit{mod}\ 2)\ xs))\ \mathit{div}\ 2)$$

xs is a solution iff...

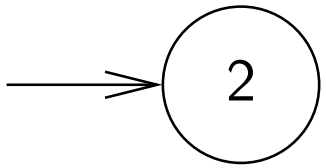
- ...it is a solution modulo 2, and...
- ...quotient of xs and 2 is a solution of another equation with same coefficients, but different right-hand side.

Reachable right-hand sides are bounded

$$\mathit{If}\ |m| \leq \mathit{max}\ |l|\ (\sum k{\leftarrow}ks.\ |k|)\ \ \mathit{then}$$
$$|(m - \mathit{eval\text{-}dioph}\ ks\ (\mathit{map}\ (\lambda x.\ x\ \mathit{mod}\ 2)\ xs))\ \mathit{div}\ 2|$$
$$\leq \mathit{max}\ |l|\ (\sum k{\leftarrow}ks.\ |k|).$$

Diophantine Equations — Example

Formula: $2x - 3y = 2$



Diophantine Equations — Example

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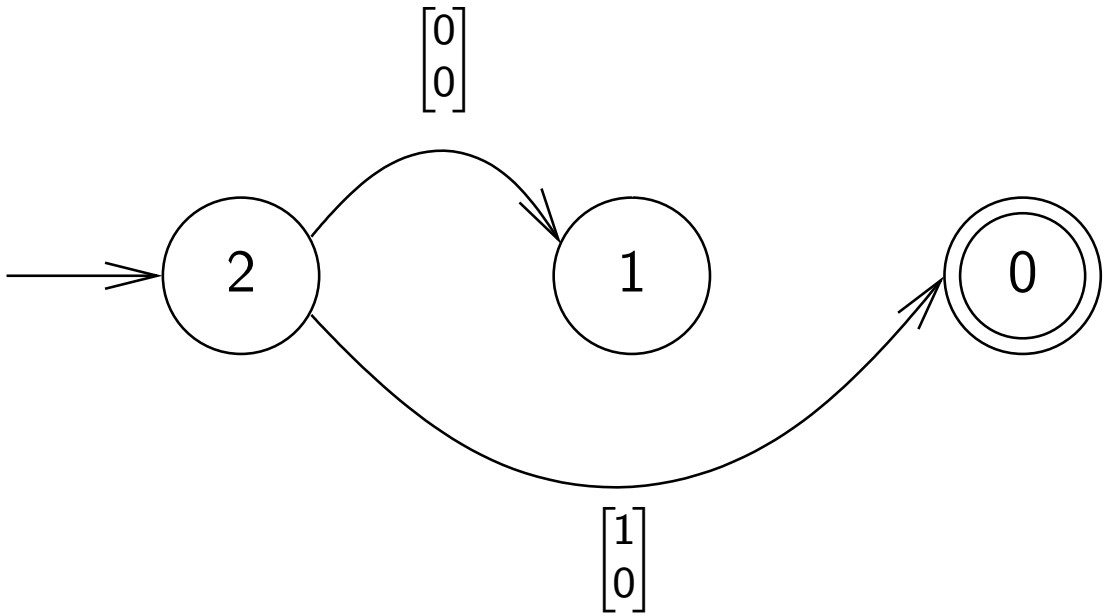
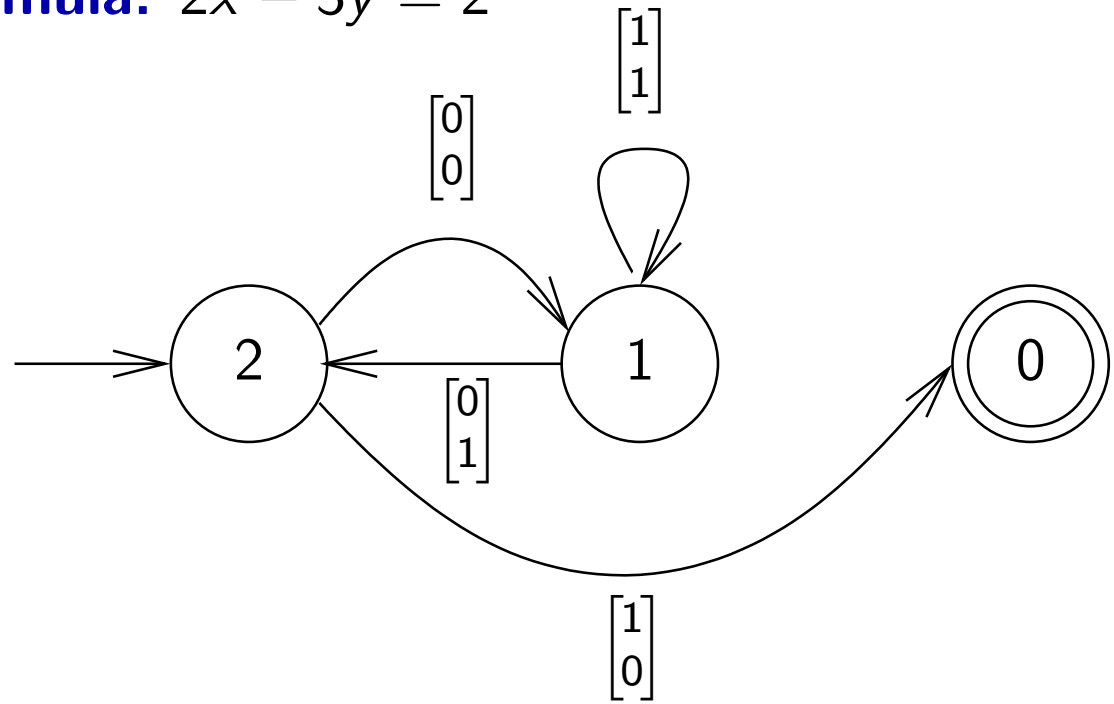


Reachable states

$(2 - (2 * 0 - 3 * 0))\ div\ 2 = 1$
$(2 - (2 * 1 - 3 * 0))\ div\ 2 = 0$
$(2 * 0 - 3 * 1)\ mod\ 2 = 1$
$(2 * 1 - 3 * 1)\ mod\ 2 = 1$

Diophantine Equations — Example

Formula: $2x - 3y = 2$

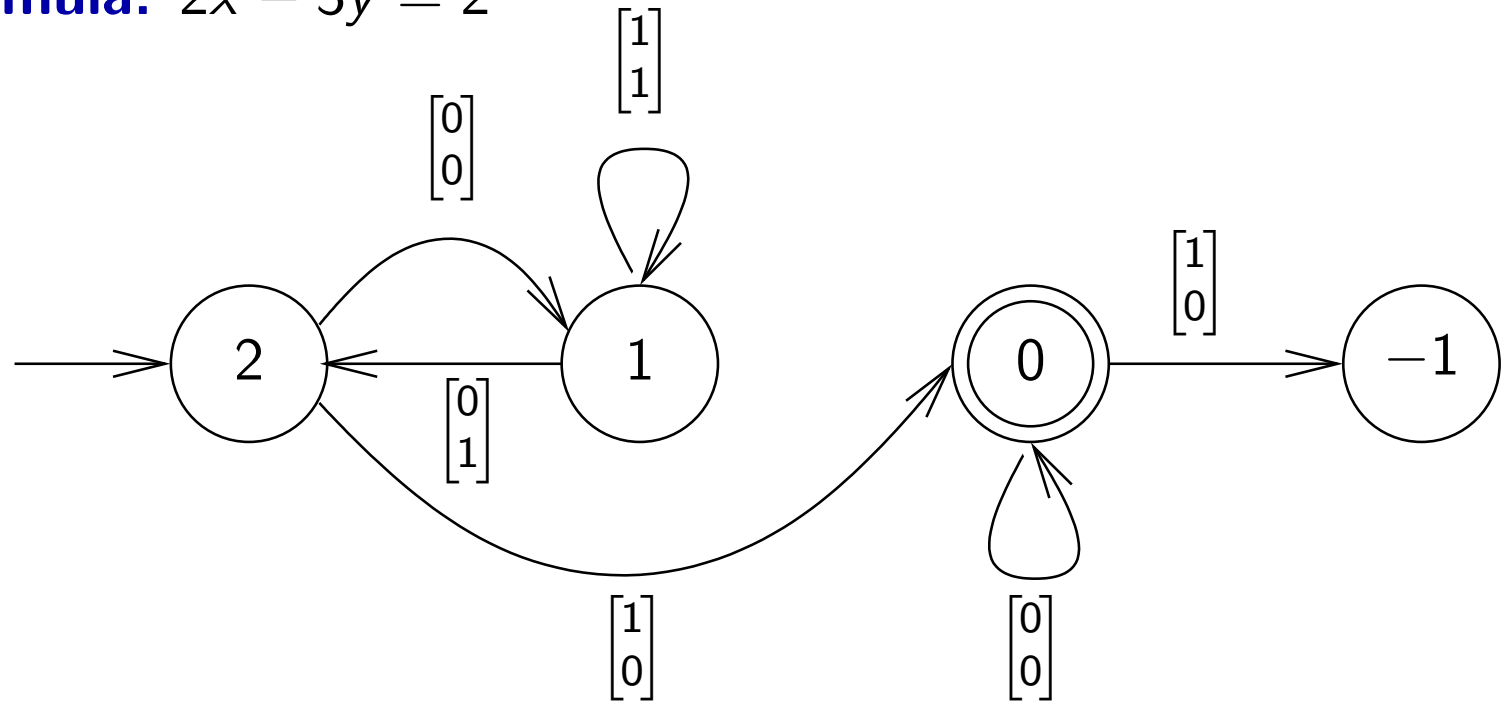


Reachable states

$(1 - (2 * 0 - 3 * 1))\ div\ 2 = 2$
$(1 - (2 * 1 - 3 * 1))\ div\ 2 = 1$
$(2 * 0 - 3 * 0)\ mod\ 2 = 0$
$(2 * 1 - 3 * 0)\ mod\ 2 = 0$

Diophantine Equations — Example

Formula: $2x - 3y = 2$

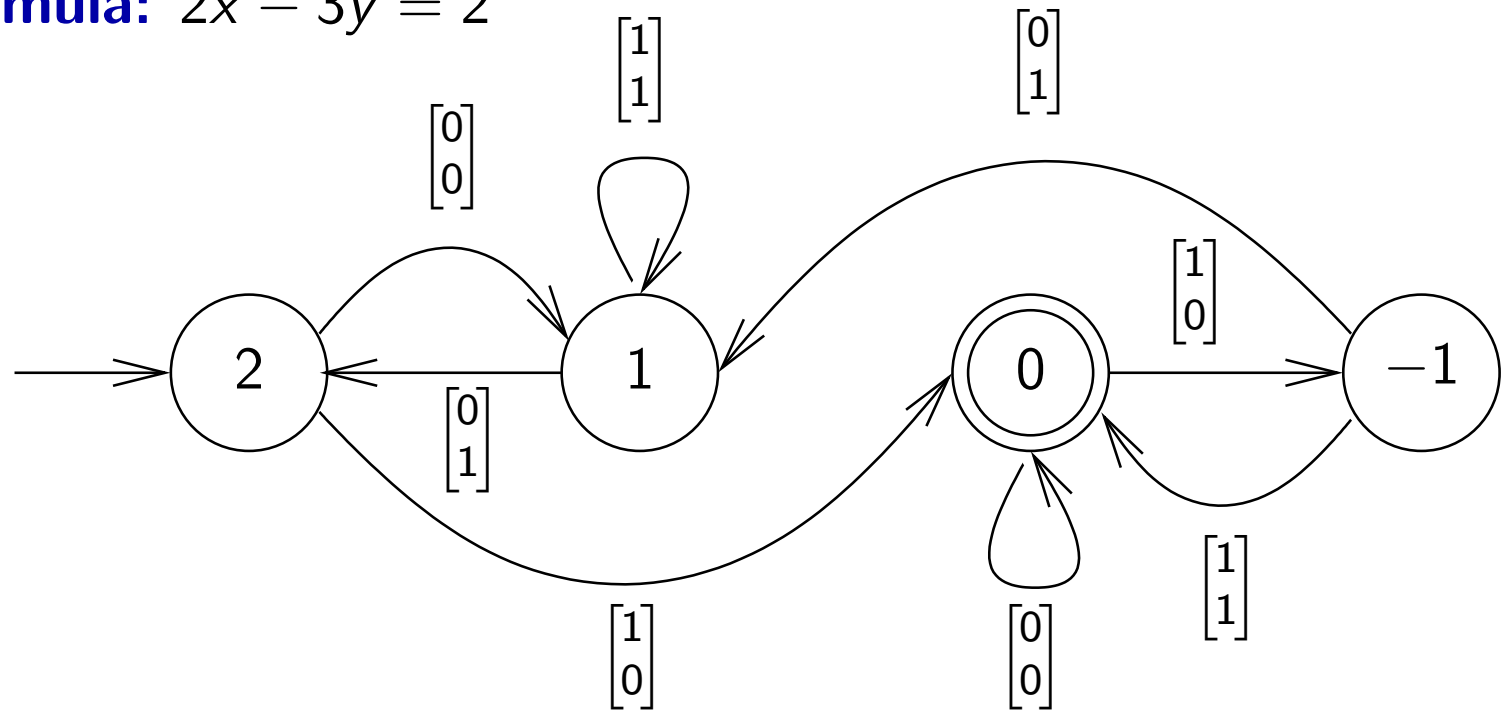


Reachable states

$(0 - (2 * 0 - 3 * 0))\ div\ 2 = 0$
$(0 - (2 * 1 - 3 * 0))\ div\ 2 = -1$
$(2 * 0 - 3 * 1)\ mod\ 2 = 1$
$(2 * 1 - 3 * 1)\ mod\ 2 = 1$

Diophantine Equations — Example

Formula: $2x - 3y = 2$

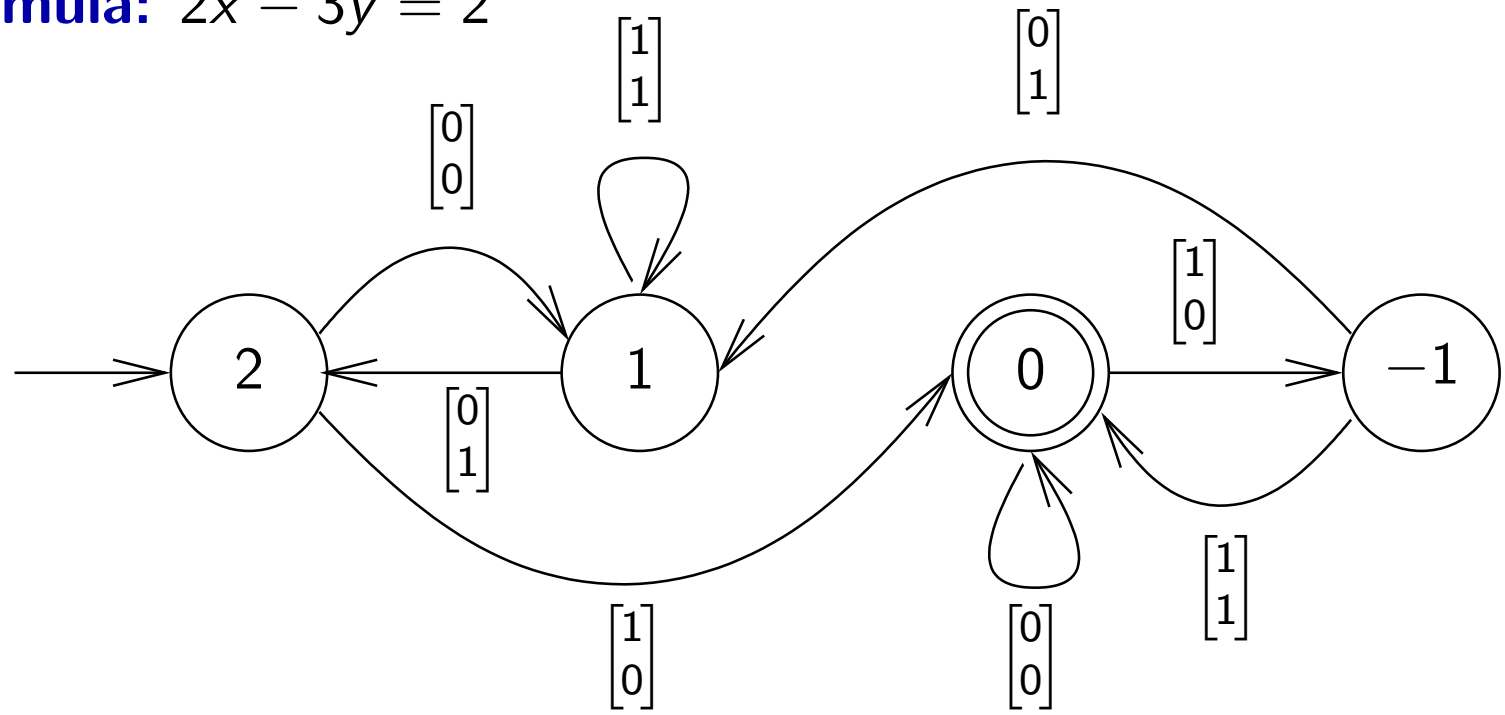


Reachable states

$(-1 - (2 * 0 - 3 * 1))\ div\ 2 = 1$
$(-1 - (2 * 1 - 3 * 1))\ div\ 2 = 0$
$(2 * 0 - 3 * 0)\ mod\ 2 = 0$
$(2 * 1 - 3 * 0)\ mod\ 2 = 0$

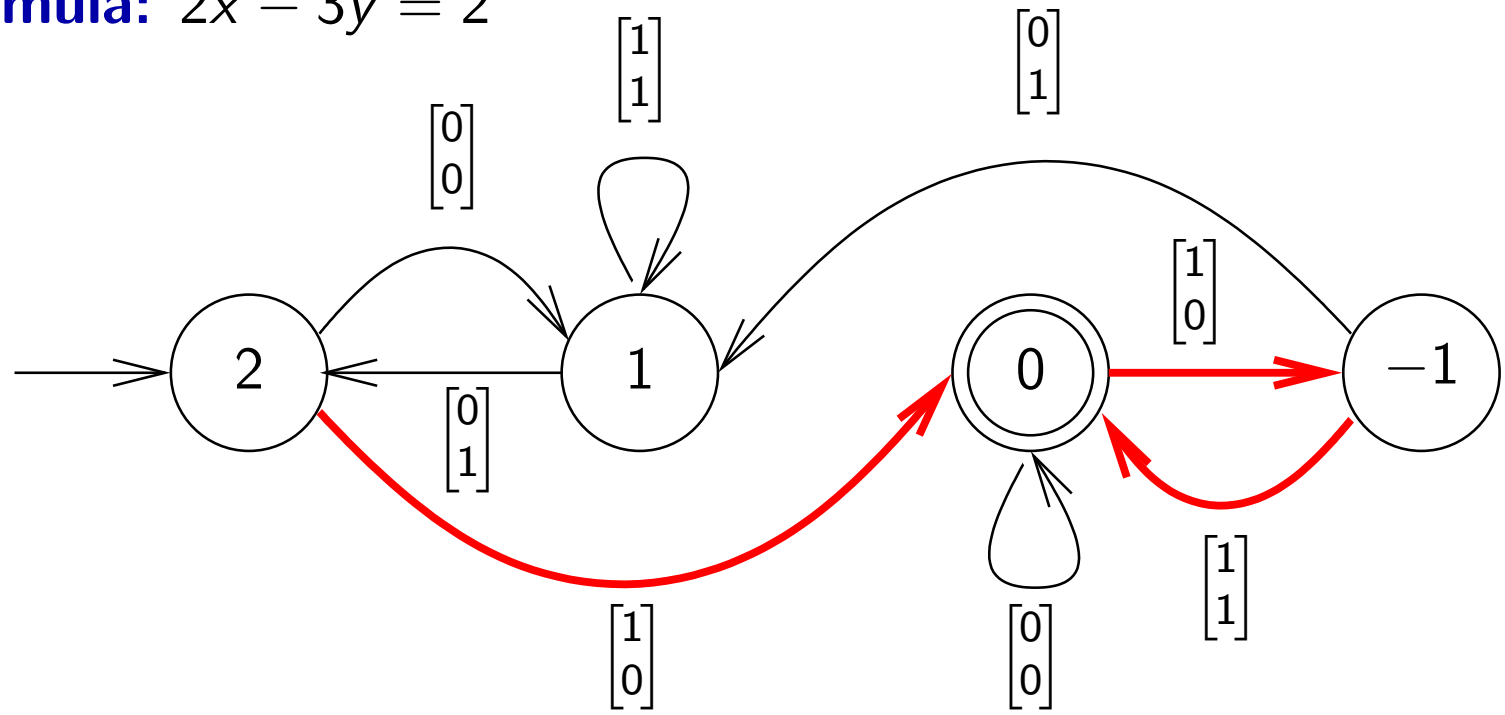
Diophantine Equations — Example

Formula: $2x - 3y = 2$



Diophantine Equations — Example

Formula: $2x - 3y = 2$

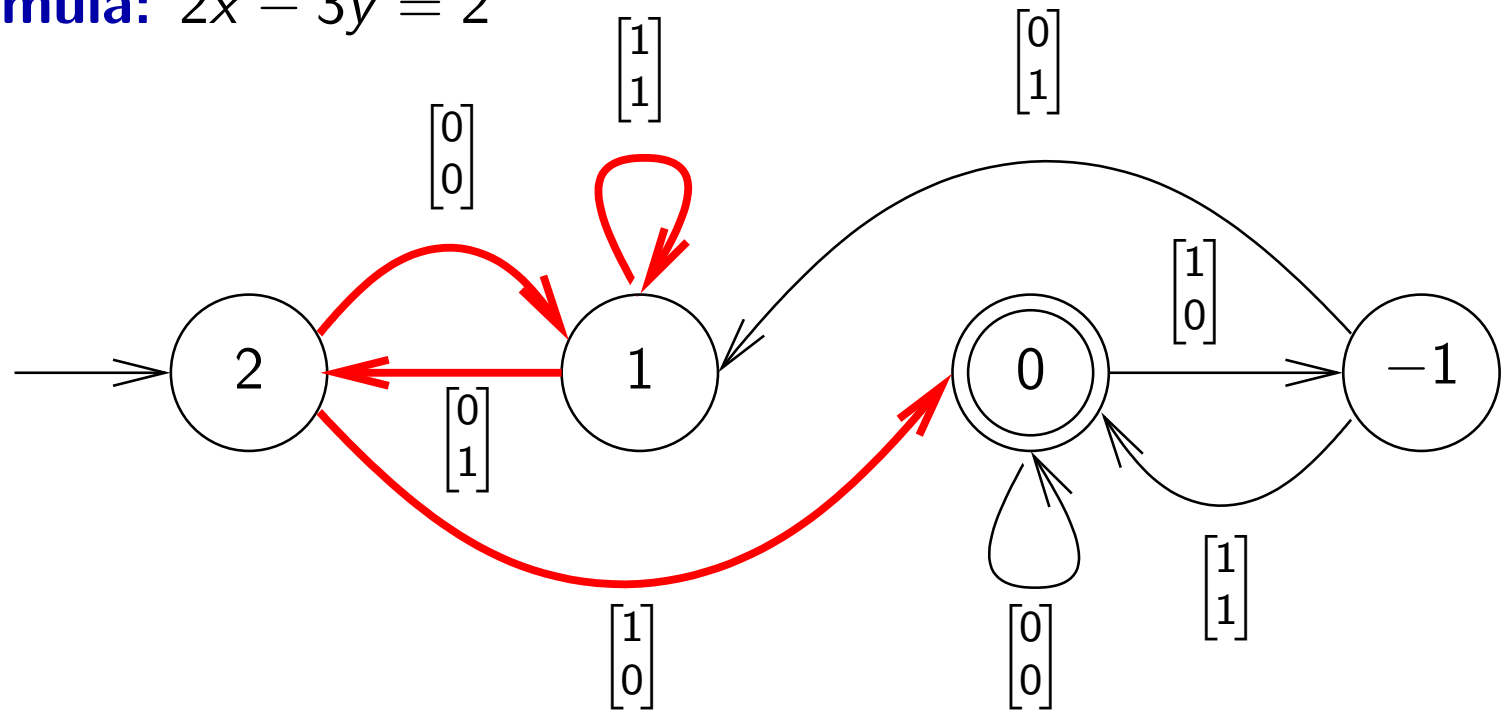


Some solutions

1	2	4	8	16	32	64	128	
1	1	1						$7 \cdot 2 = 14$
0	0	1						$4 \cdot 3 = 12$

Diophantine Equations — Example

Formula: $2x - 3y = 2$

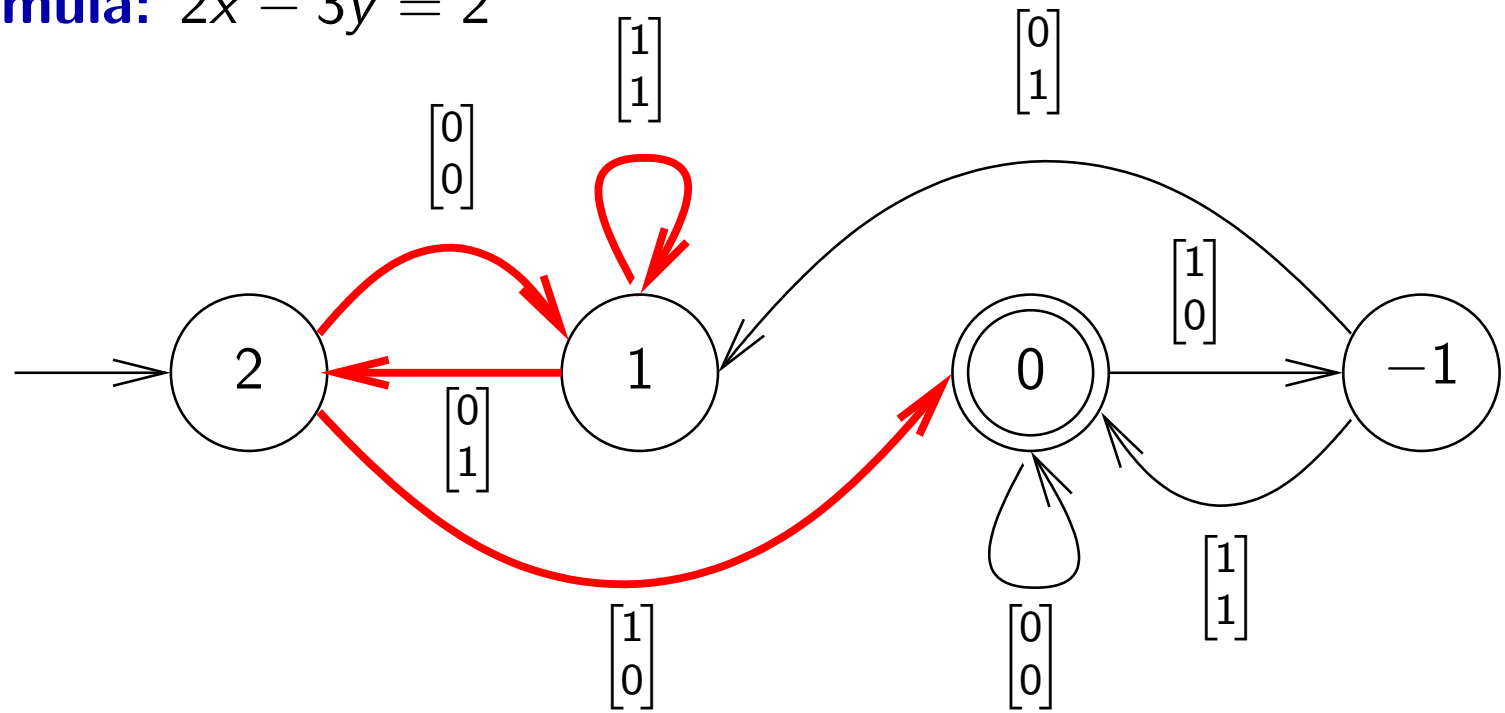


Some solutions

1	2	4	8	16	32	64	128	
0	1	1	0	1				$22 \cdot 2 = 44$
0	1	1	1	0				$14 \cdot 3 = 42$

Diophantine Equations — Example

Formula: $2x - 3y = 2$

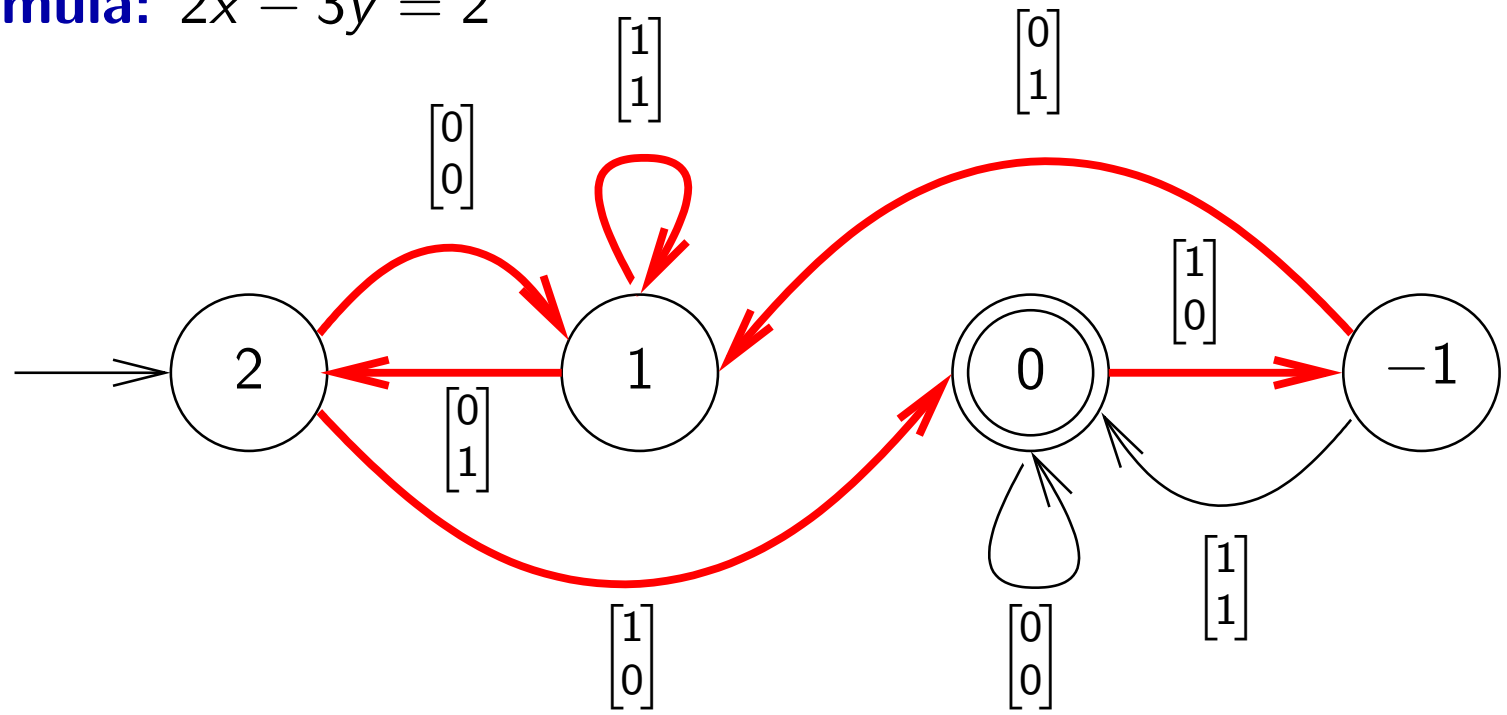


Some solutions

1	2	4	8	16	32	64	128	
0	1	0	1					$10 \cdot 2 = 20$
0	1	1	0					$6 \cdot 3 = 18$

Diophantine Equations — Example

Formula: $2x - 3y = 2$



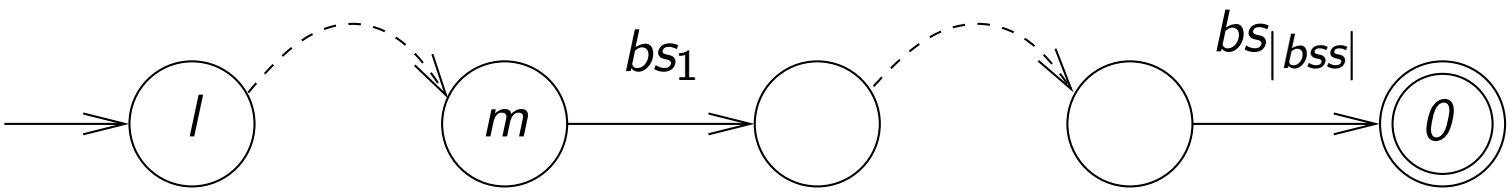
Some solutions

1	2	4	8	16	32	64	128			
0	1	0	1	1	0	0	1	154	$\cdot 2 =$	308
0	1	1	0	0	1	1	0	102	$\cdot 3 =$	306

Diophantine Equations — The Code

```
eq-dfa :: nat ⇒ int list ⇒ int ⇒ dfa
eq-dfa n ks l ≡
let (is, js) = dioph-dfs n ks l
in (map (λj. make-bdd
              (λxs. if eval-dioph ks xs mod 2 = j mod 2
                    then the is[(|(j − eval-dioph ks xs) div 2|)nat]
                    else |js|)
              n [])
     js @
    [Leaf |js|],
    map (λj. j = 0) js @ [False])
```

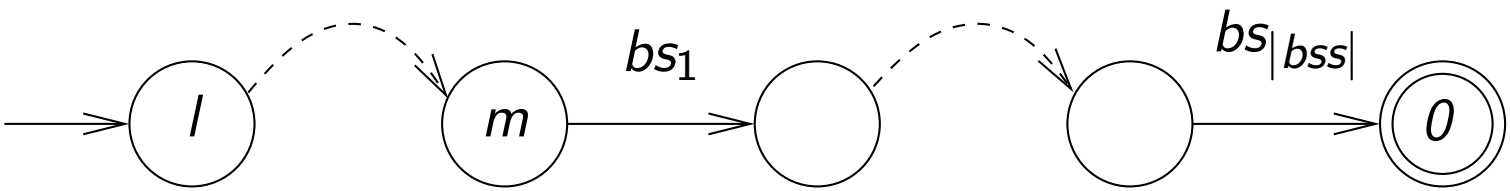
Diophantine Equations — Correctness



Strengthened statement

If $(l, m) \in (\mathit{dioph\text{-}succs}\ n\ ks)^*$ *and* $\forall\, bs{\in}bss.\ |bs| = n$ *then*
$\mathit{dfa\text{-}accepting}\ (\mathit{eq\text{-}dfa}\ n\ ks\ l)$
$(\mathit{dfa\text{-}steps}\ (\mathit{eq\text{-}dfa}\ n\ ks\ l)$
$(\mathit{the}\ (\mathit{fst}\ (\mathit{dioph\text{-}dfs}\ n\ ks\ l))[(\!|m|\!)_{nat}])\ bss) =$
$(\mathit{eval\text{-}dioph}\ ks\ (\mathit{nats\text{-}of\text{-}boolss}\ n\ bss) = m)$.

Diophantine Equations — Correctness



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$\mathit{dfa\text{-}accepting}\ (\mathit{eq\text{-}dfa}\ n\ ks\ l)$
$(\mathit{dfa\text{-}steps}\ (\mathit{eq\text{-}dfa}\ n\ ks\ l)$
$(\mathit{the}\ (\mathit{fst}\ (\mathit{dioph\text{-}dfs}\ n\ ks\ l))[(\!|m|\!)_{nat}])\ bss) =$
$(\mathit{eval\text{-}dioph}\ ks\ (\mathit{nats\text{-}of\text{-}boolss}\ n\ bss) = m)$.

Corollary ($l = m$)

If $\forall\, bs{\in}bss.\ |bs| = n$ *then*
$\mathit{dfa\text{-}accepts}\ (\mathit{eq\text{-}dfa}\ n\ ks\ l)\ bss =$
$(\mathit{eval\text{-}dioph}\ ks\ (\mathit{nats\text{-}of\text{-}boolss}\ n\ bss) = l)$.

Diophantine Inequations

Method by Boudet and Comon

$$(\mathit{eval\text{-}dioph}\ \mathit{ks}\ \mathit{xs} \leq l) =$$
$$(\mathit{eval\text{-}dioph}\ \mathit{ks}\ (\mathit{map}\ (\lambda x.\ x\ \mathit{div}\ 2)\ \mathit{xs}) \leq$$
$$(l - \mathit{eval\text{-}dioph}\ \mathit{ks}\ (\mathit{map}\ (\lambda x.\ x\ \mathit{mod}\ 2)\ \mathit{xs}))\ \mathit{div}\ 2)$$

Diophantine Inequations

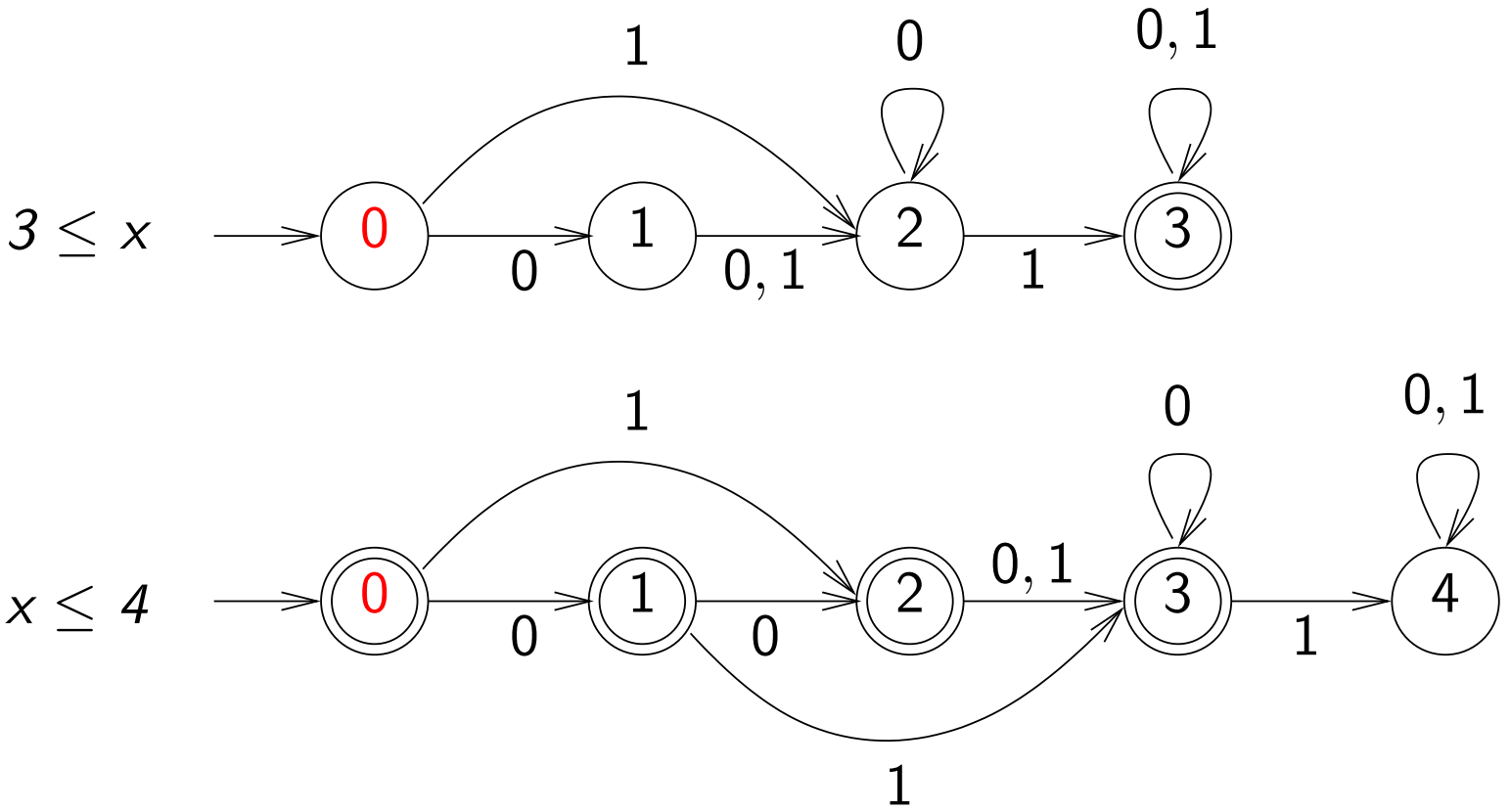
Method by Boudet and Comon

$(\mathit{eval\text{-}dioph}\ \mathit{ks}\ \mathit{xs} \leq \mathit{l}) =$
$(\mathit{eval\text{-}dioph}\ \mathit{ks}\ (\mathit{map}\ (\lambda x.\ x\ \mathit{div}\ 2)\ \mathit{xs}) \leq$
$(\mathit{l} - \mathit{eval\text{-}dioph}\ \mathit{ks}\ (\mathit{map}\ (\lambda x.\ x\ \mathit{mod}\ 2)\ \mathit{xs}))\ \mathit{div}\ 2)$

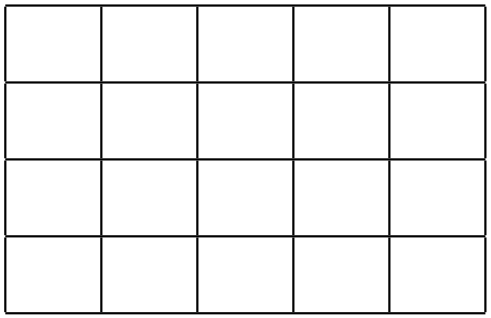
The Code

$\mathit{ineq\text{-}dfa} :: \mathit{nat} \Rightarrow \mathit{int\ list} \Rightarrow \mathit{int} \Rightarrow \mathit{dfa}$
$\mathit{ineq\text{-}dfa}\ n\ \mathit{ks}\ \mathit{l} \equiv$
$\mathit{let}\ (\mathit{is}, \mathit{js}) = \mathit{dioph\text{-}ineq\text{-}dfs}\ n\ \mathit{ks}\ \mathit{l}$
$\mathit{in}\ (\mathit{map}\ (\lambda j.\ \mathit{make\text{-}bdd}$
$(\lambda \mathit{xs}.\ \mathit{the}\ \mathit{is}[(\!|(j - \mathit{eval\text{-}dioph}\ \mathit{ks}\ \mathit{xs})\ \mathit{div}\ 2|\!)_{nat}])\ n$
$[])$
$\mathit{js},$
$\mathit{map}\ (\lambda j.\ 0 \leq j)\ \mathit{js})$

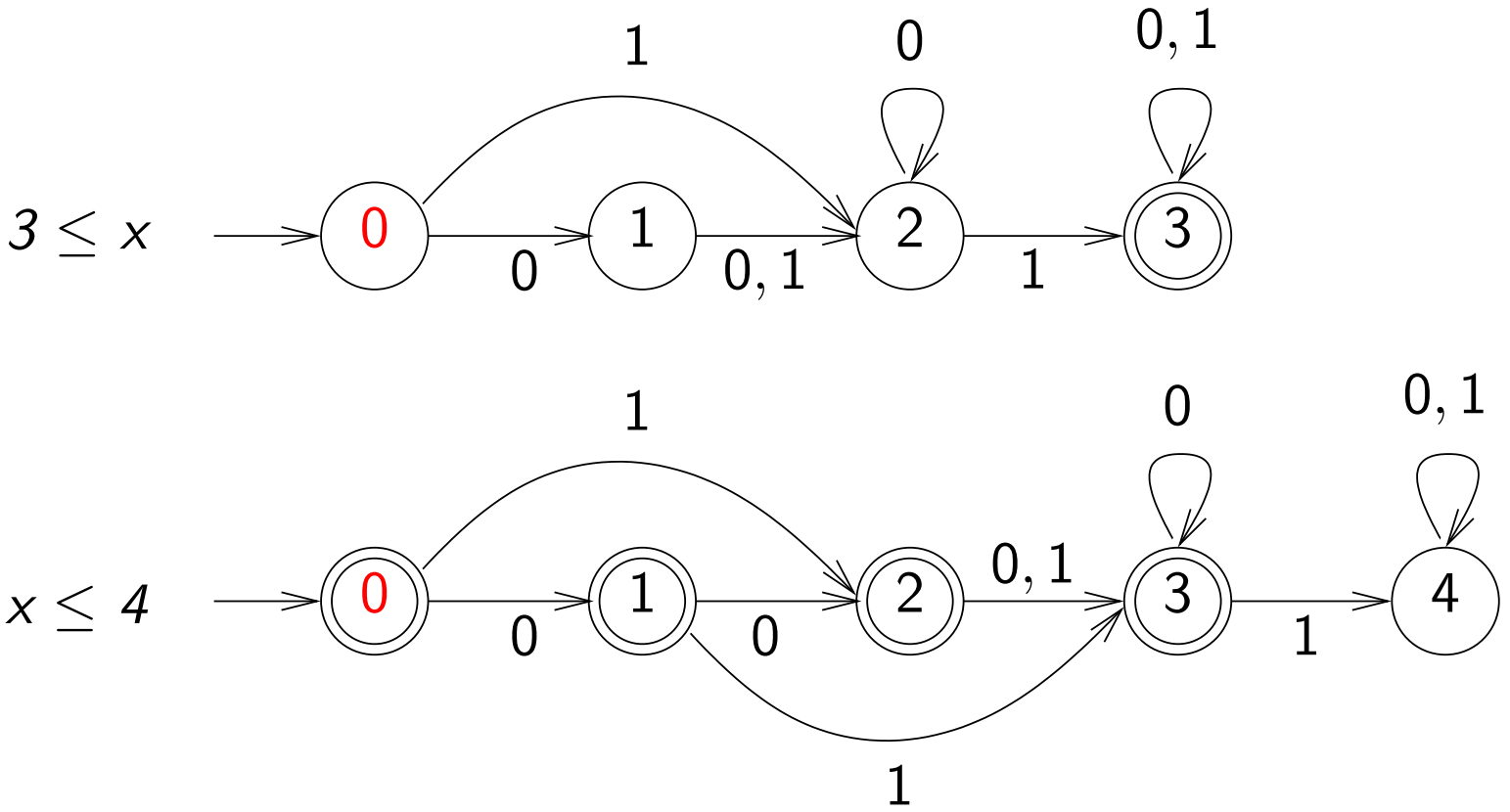
Product Automaton – Example



Todo: $(0, 0)$



Product Automaton – Example

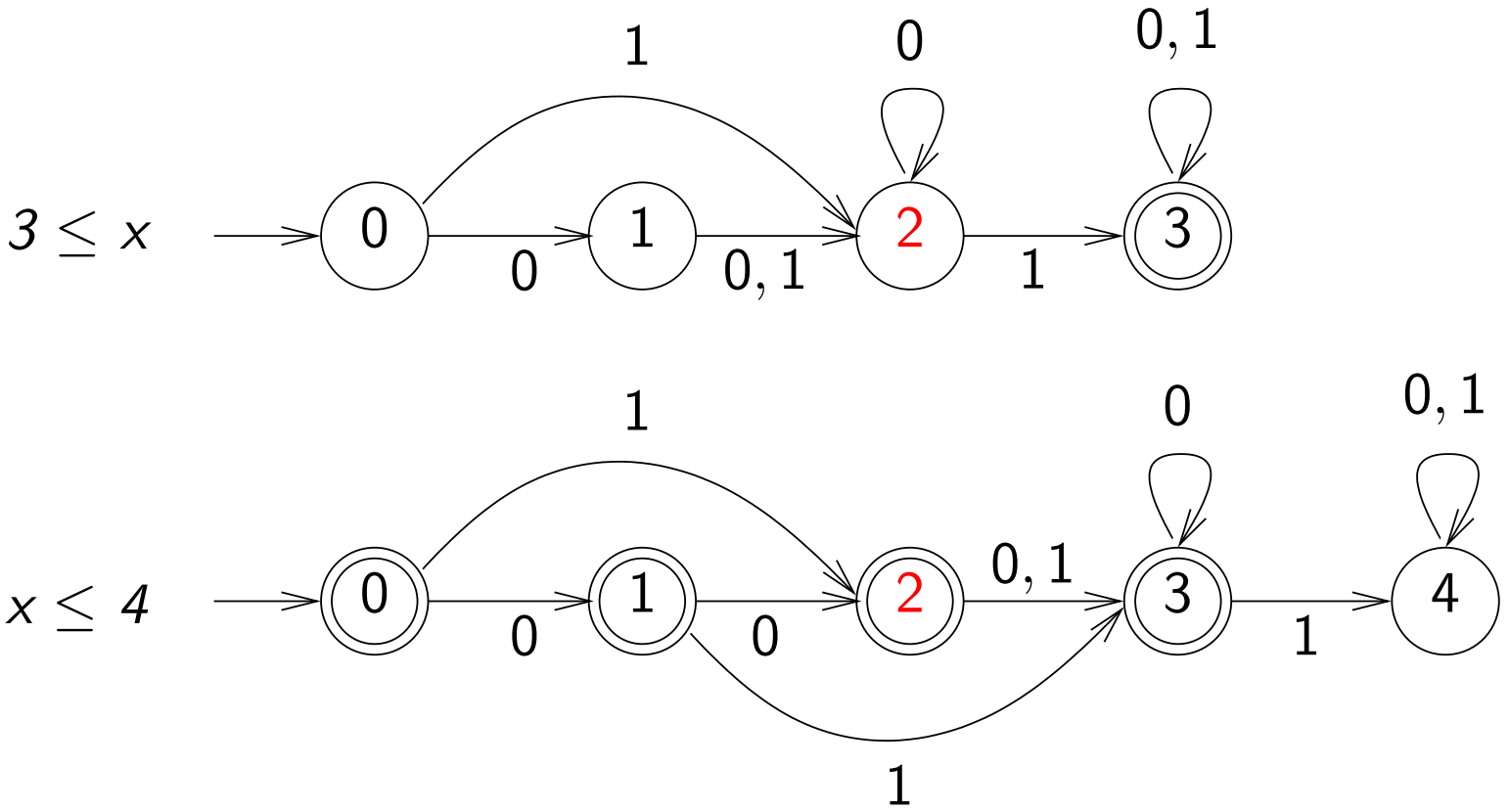


Todo: $(0, 0)$

0				

$(0, 0)$					

Product Automaton – Example

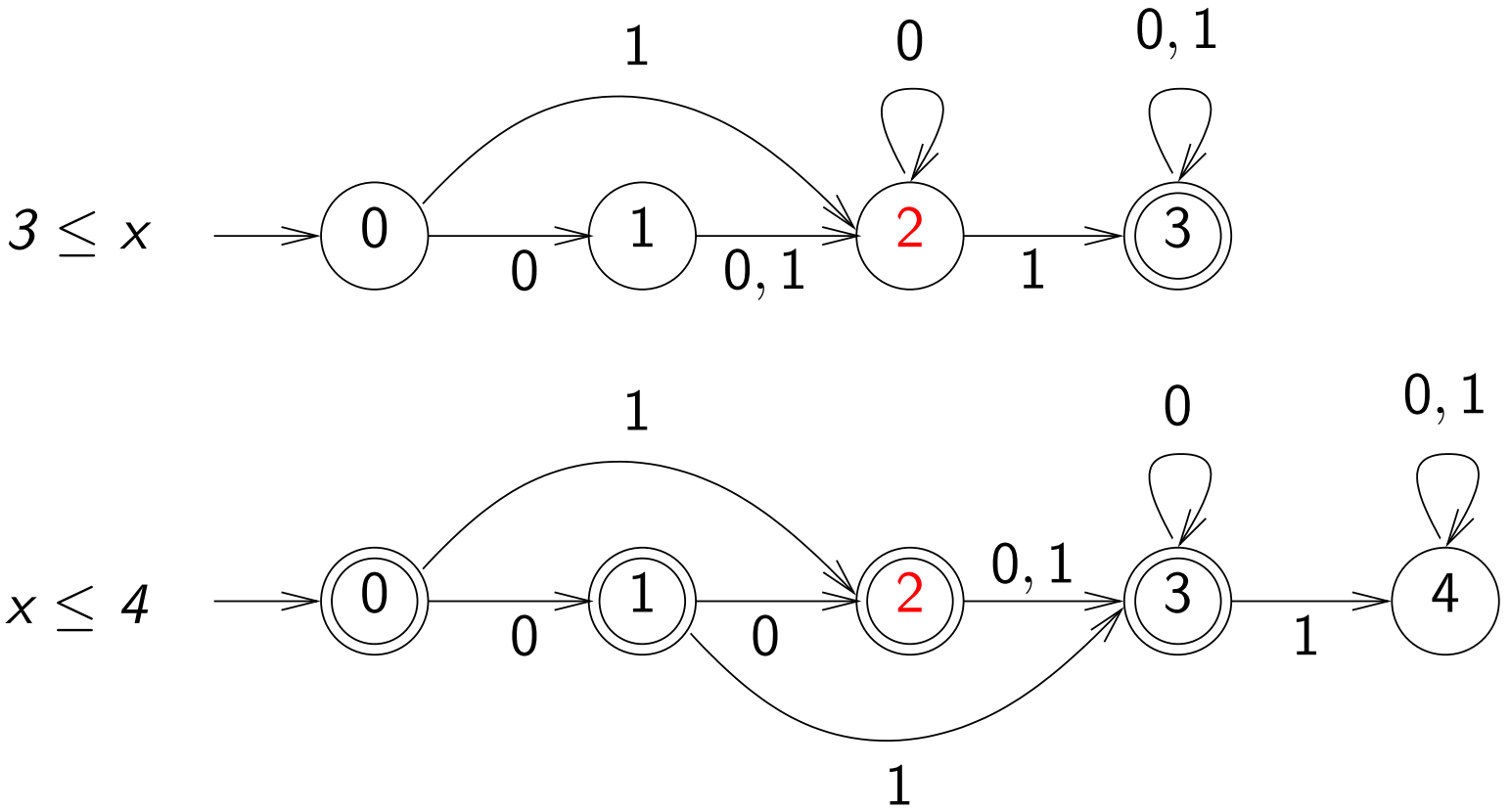


Todo: $(2,2), (1,1)$

0				

$(0,0)$					

Product Automaton – Example

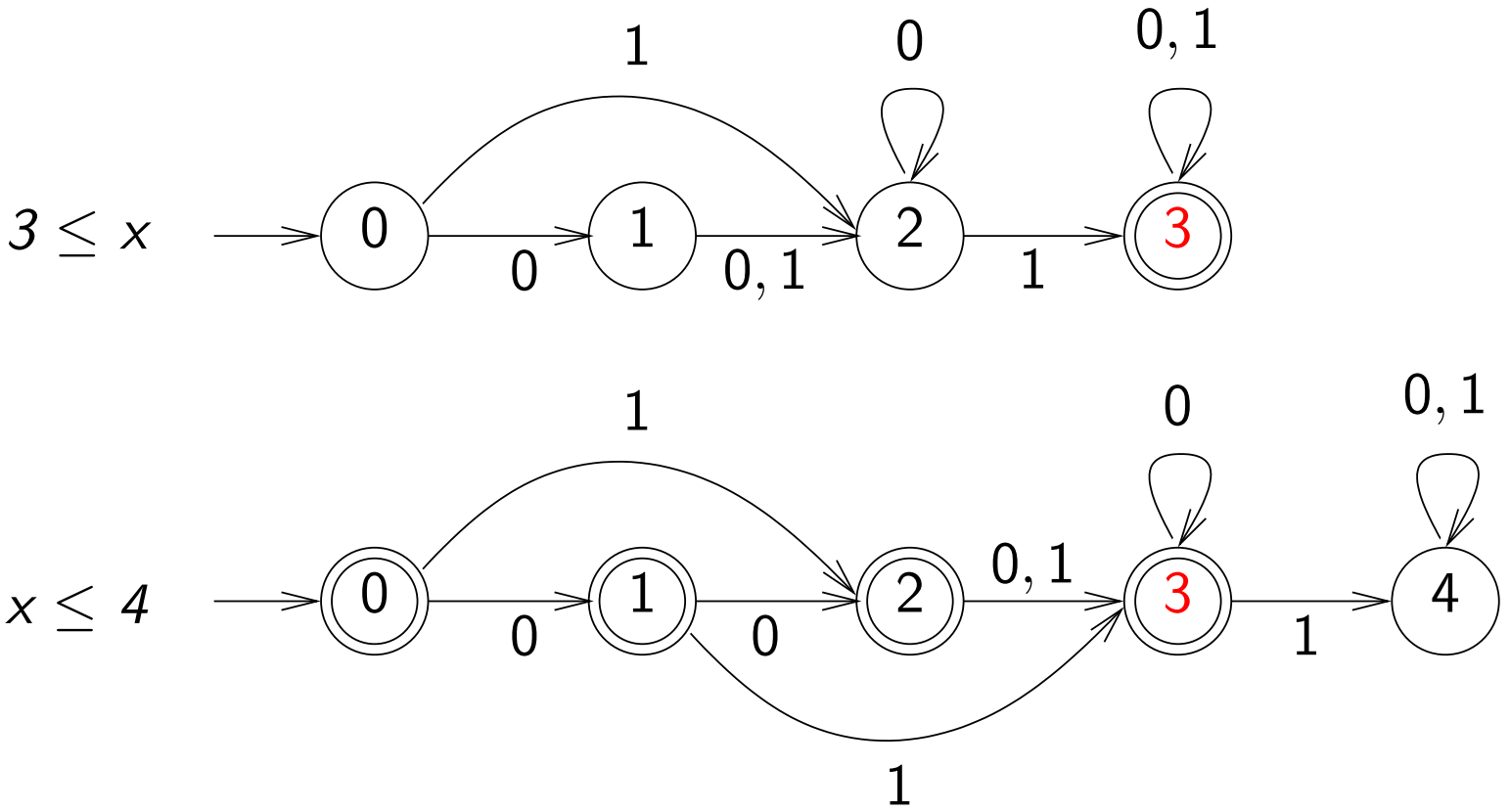


Todo: $(2,2), (1,1)$

0				
		1		

$(0,0)$	$(2,2)$				

Product Automaton – Example

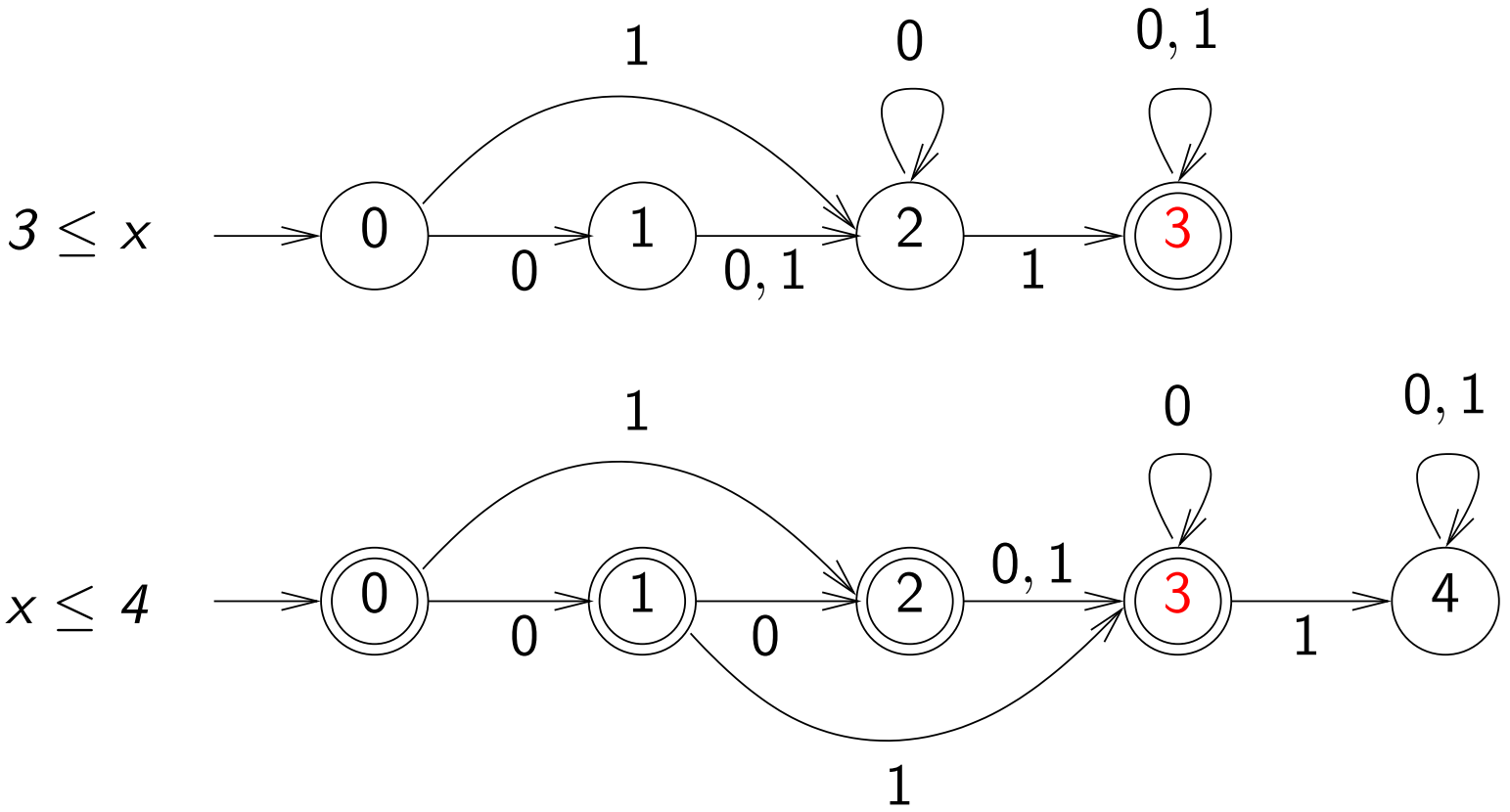


Todo: $(3,3), (2,3), (1,1)$

0				
		1		

$(0,0)$	$(2,2)$				

Product Automaton – Example

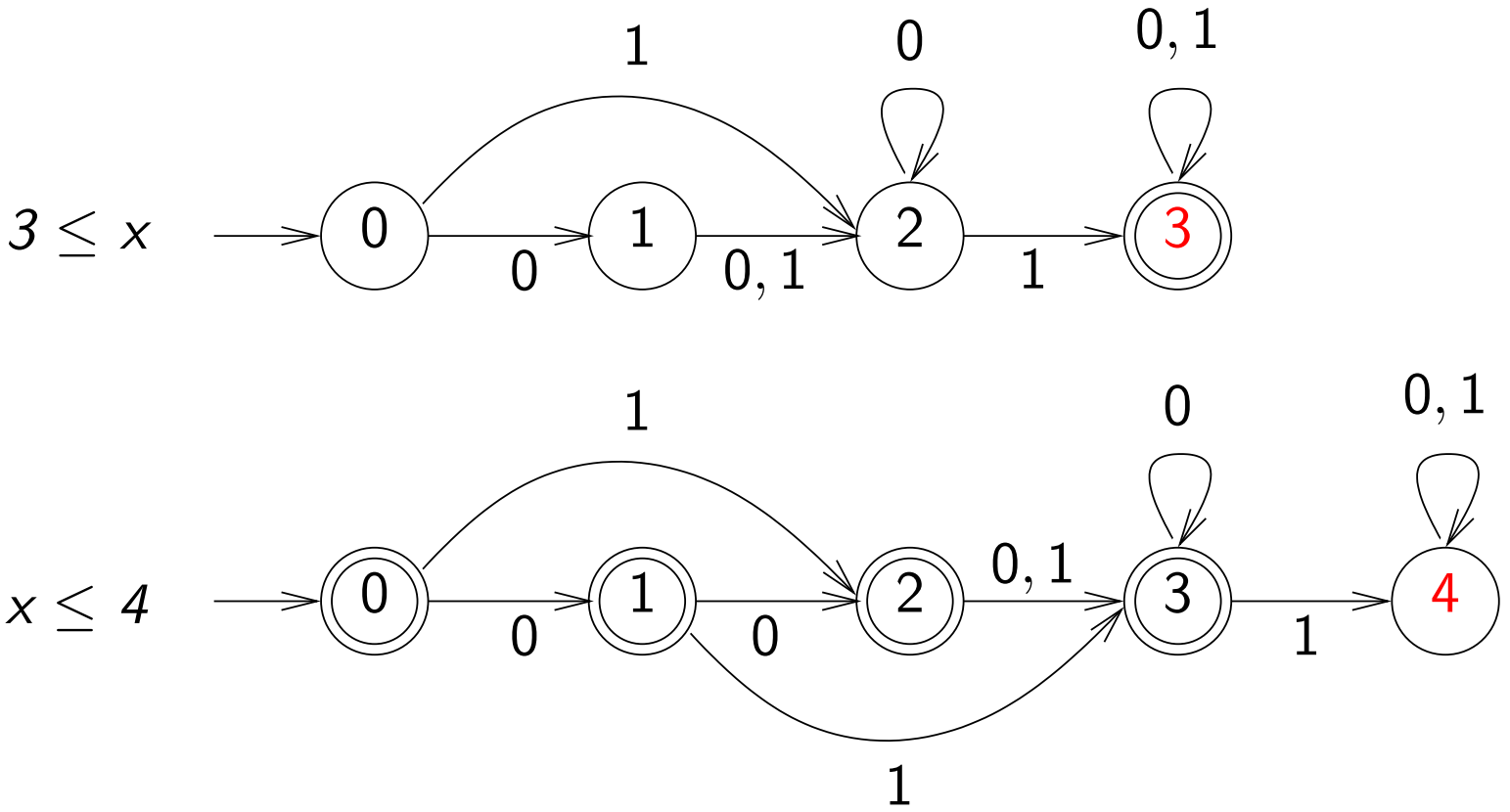


Todo: $(3,3), (2,3), (1,1)$

0				
		1		
			2	

$(0,0)$	$(2,2)$	$(3,3)$			

Product Automaton – Example

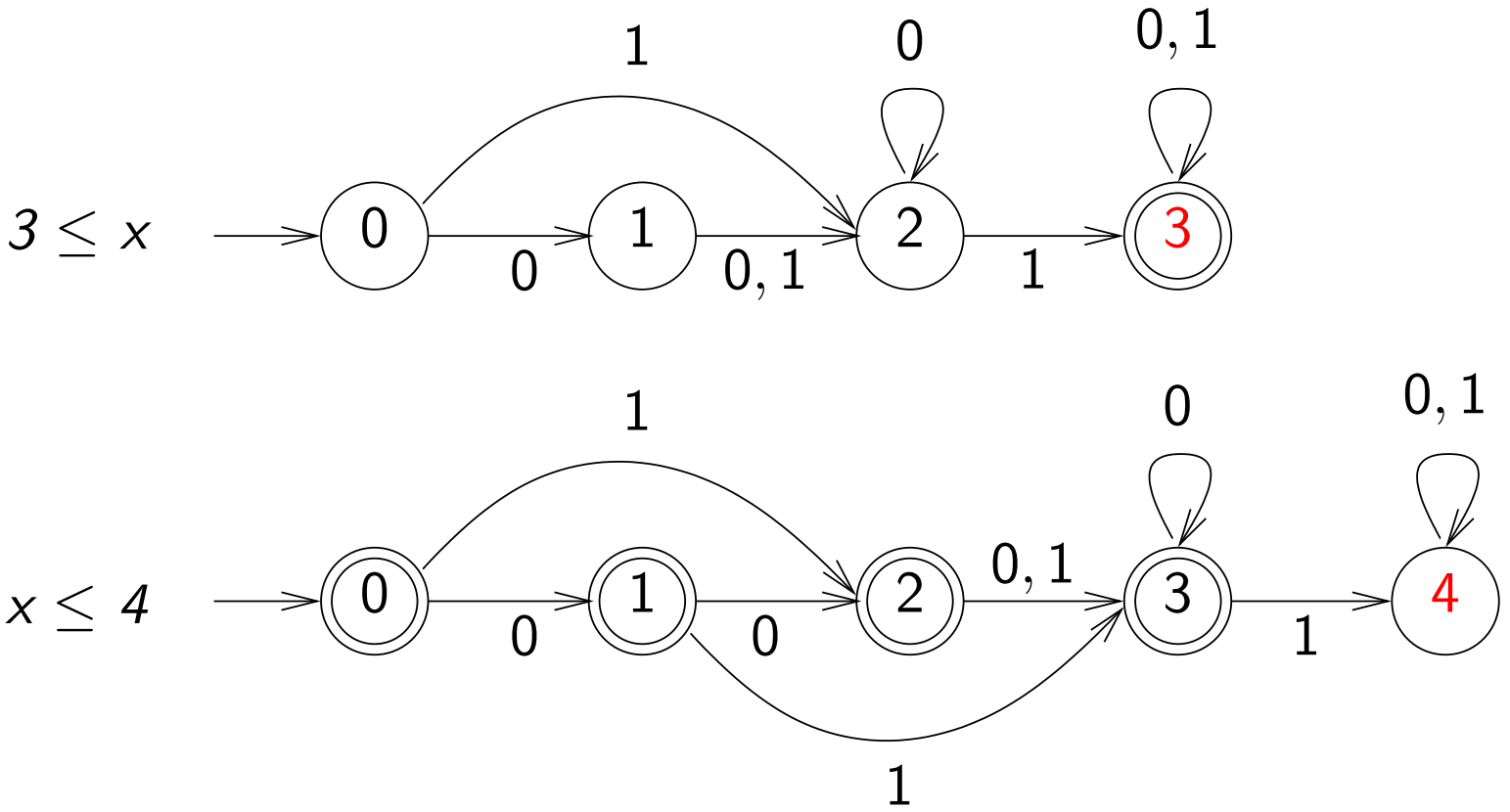


Todo: $(3,4), (3,3), (2,3), (1,1)$

0				
		1		
			2	

$(0,0)$	$(2,2)$	$(3,3)$			

Product Automaton – Example

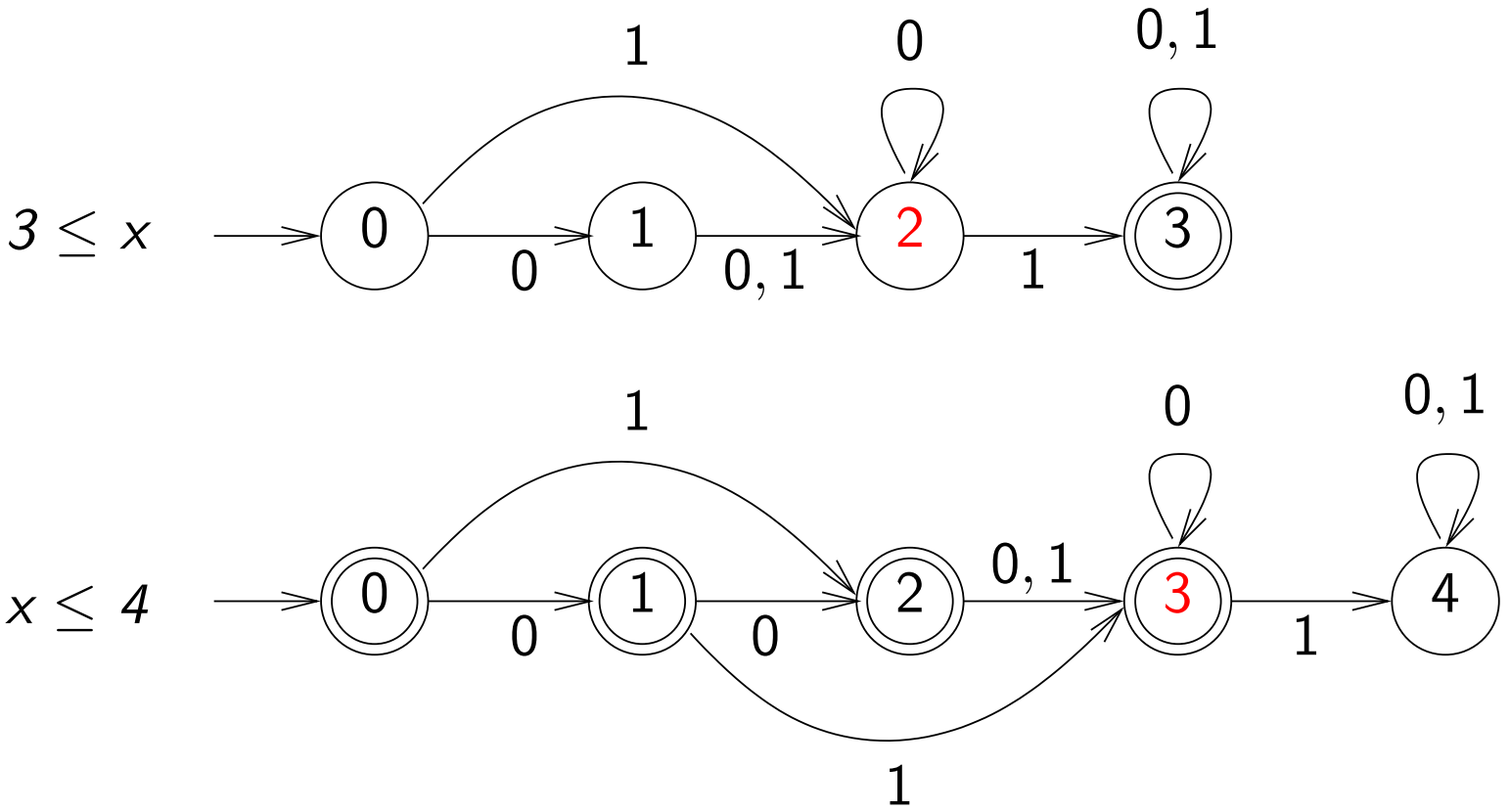


Todo: $(3,4), (3,3), (2,3), (1,1)$

0				
		1		
			2	3

$(0,0)$	$(2,2)$	$(3,3)$	$(3,4)$		

Product Automaton – Example

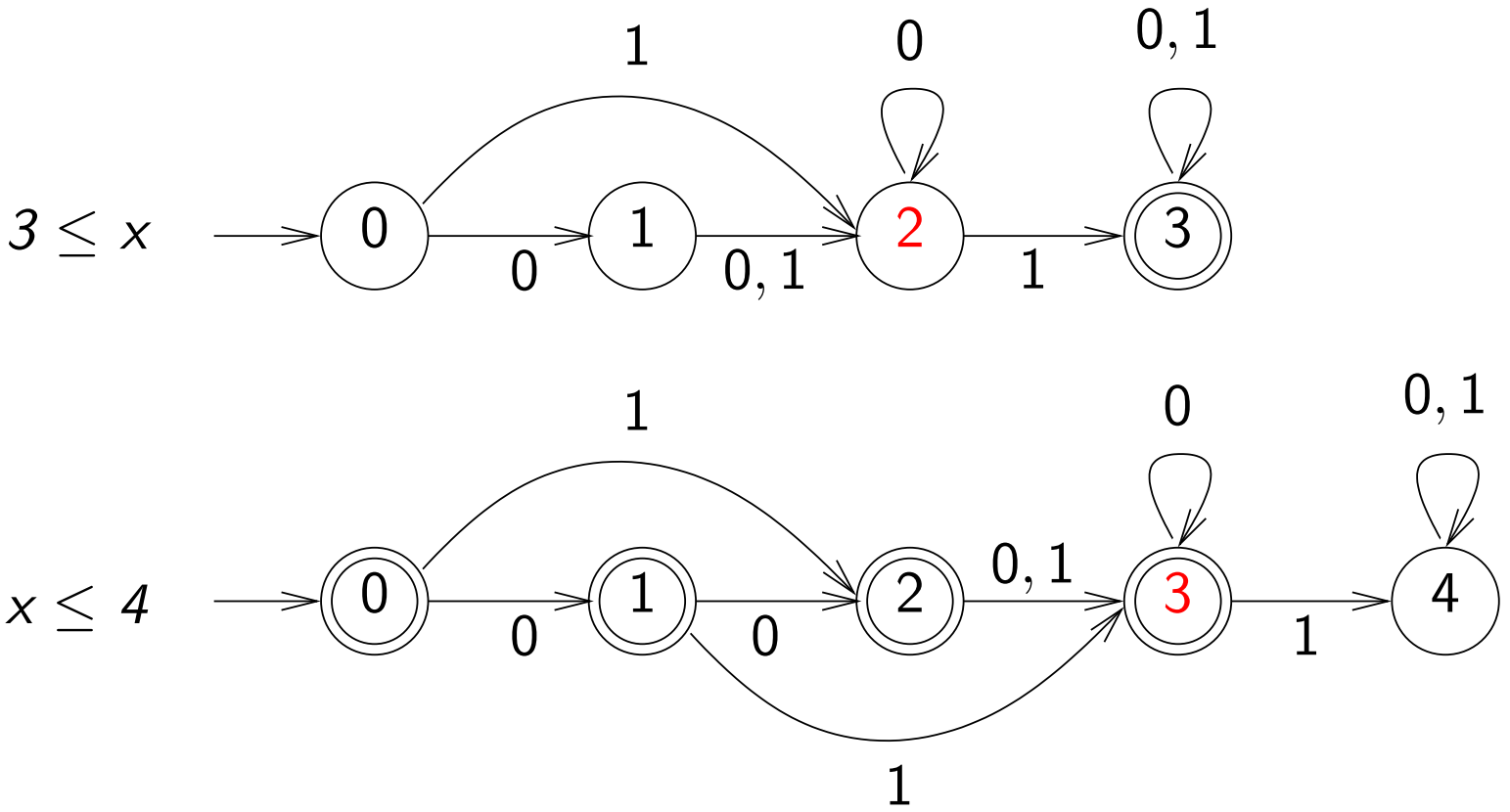


Todo: $(2,3), (1,1)$

0				
		1		
			2	3

$(0,0)$	$(2,2)$	$(3,3)$	$(3,4)$		

Product Automaton – Example

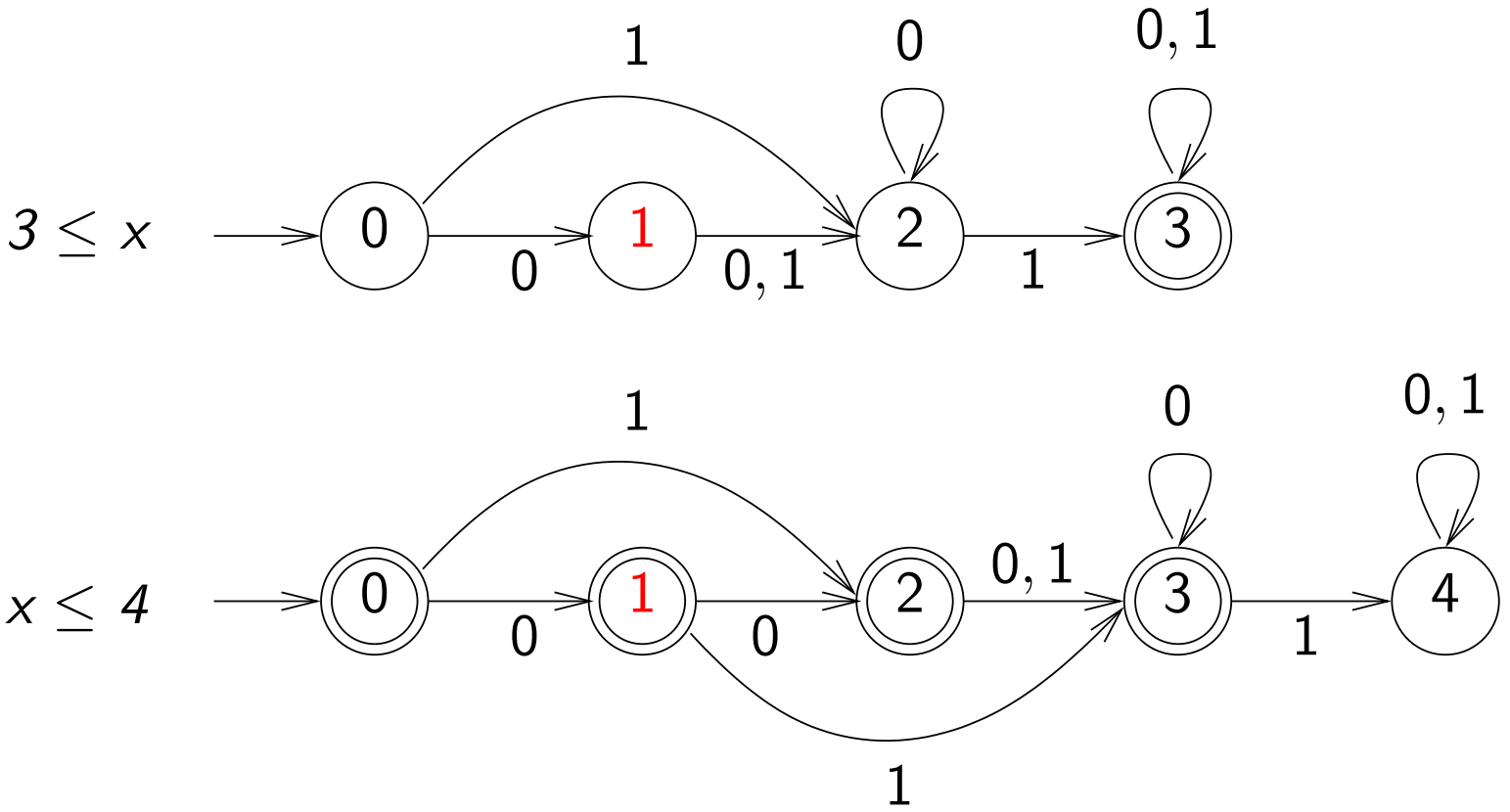


Todo: $(2,3), (1,1)$

0				
		1	4	
			2	3

$(0,0)$	$(2,2)$	$(3,3)$	$(3,4)$	$(2,3)$	

Product Automaton – Example

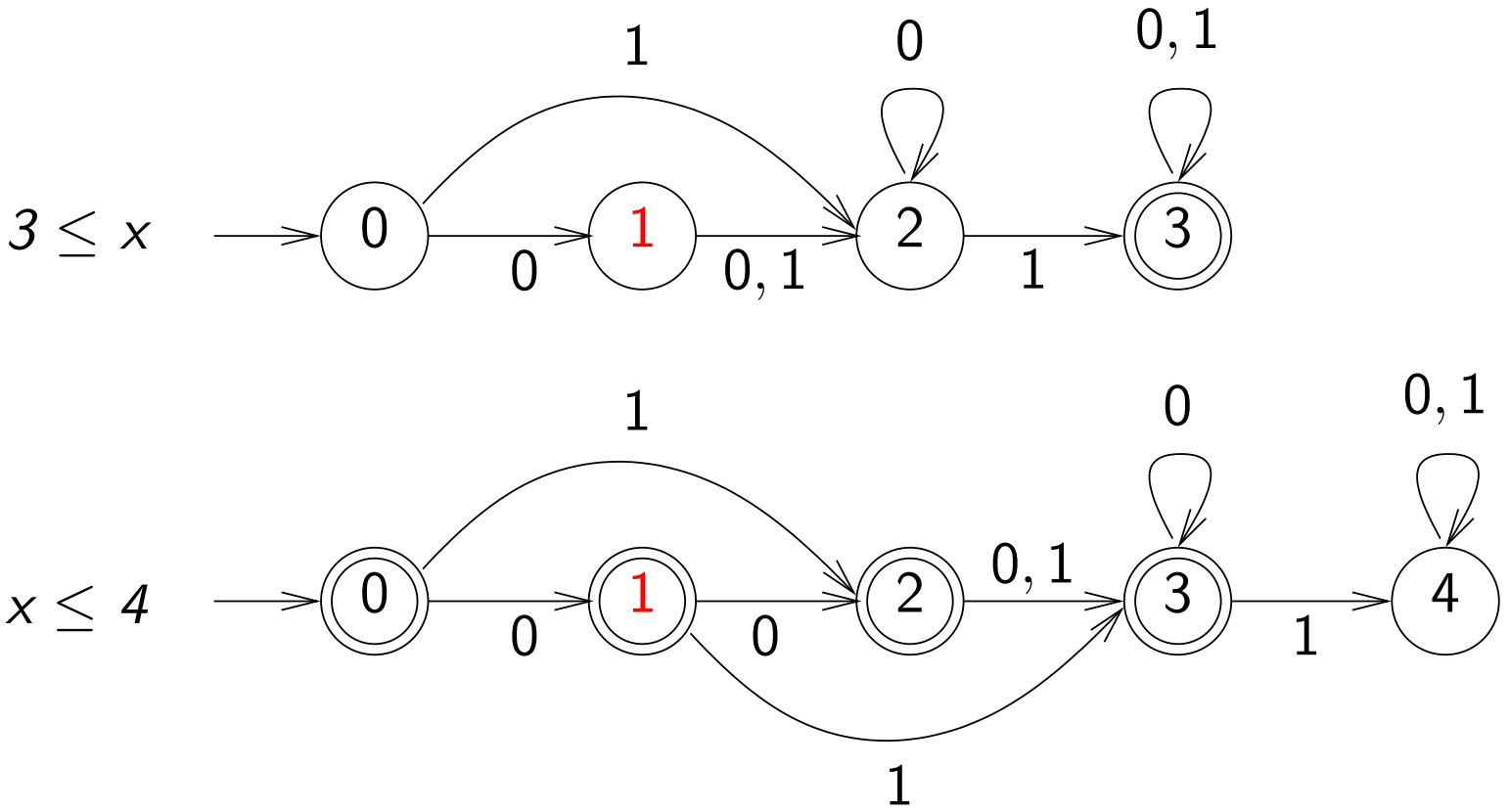


Todo: $(1, 1)$

0				
		1	4	
			2	3

$(0, 0)$	$(2, 2)$	$(3, 3)$	$(3, 4)$	$(2, 3)$	

Product Automaton – Example

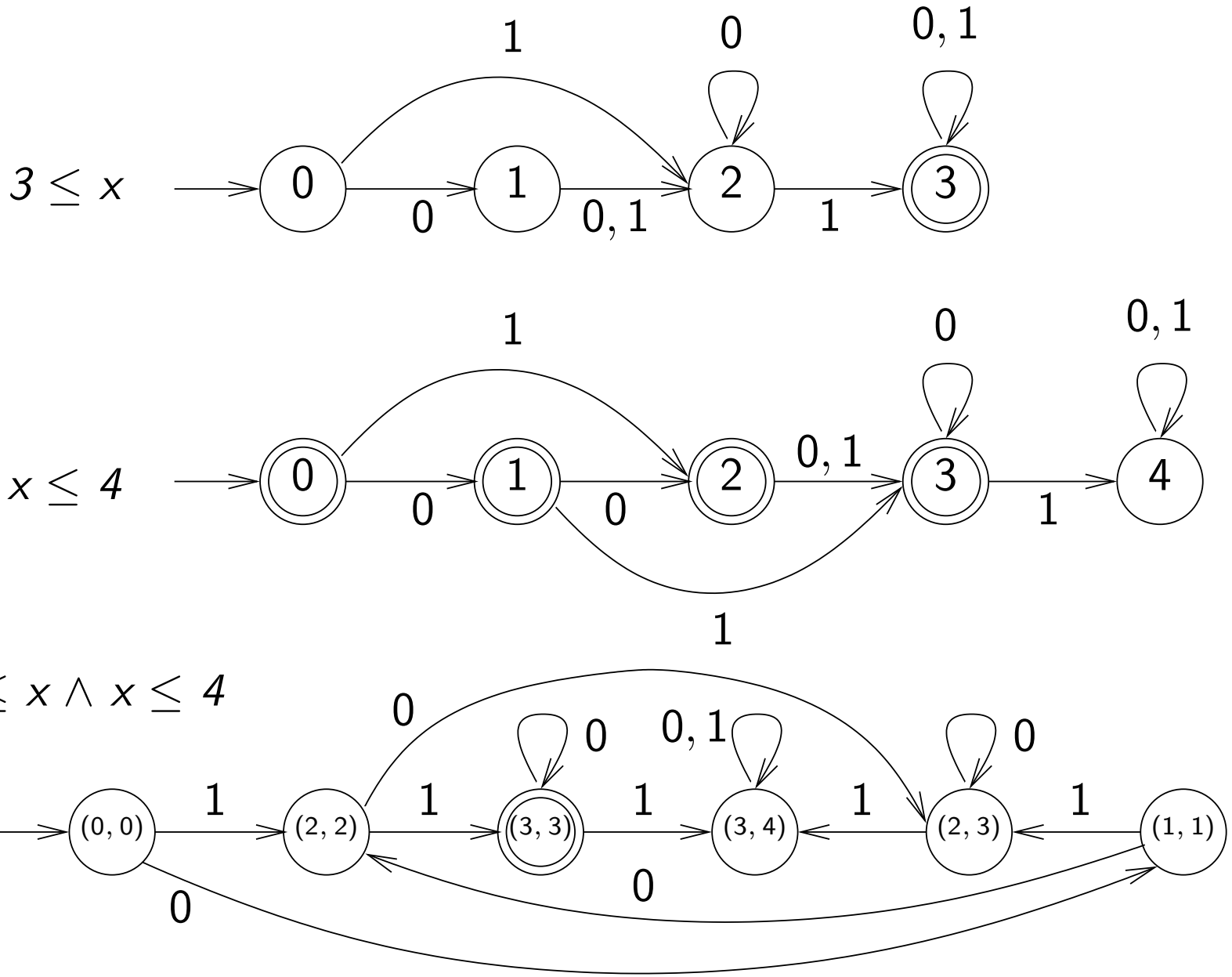


Todo: $(1, 1)$

0				
	5			
		1	4	
			2	3

$(0, 0)$	$(2, 2)$	$(3, 3)$	$(3, 4)$	$(2, 3)$	$(1, 1)$

Product Automaton – Example



Product Automaton – The Code

Building the automaton

```
binop-dfa :: (bool ⇒ bool ⇒ bool) ⇒ dfa ⇒ dfa ⇒ dfa
binop-dfa f A B ≡
let (tab, ps) = prod-dfs A B (0, 0)
in (map (λ(i, j).
              bdd-binop (λk l. the tab[k][l]) (fst A)[i] (fst B)[j])
    ps,
   map (λ(i, j). f (snd A)[i] (snd B)[j]) ps)
```

Product Automaton – The Code

Building the automaton

$$
\begin{array}{l}
\mathit{binop\text{-}dfa} :: (\mathit{bool} \Rightarrow \mathit{bool} \Rightarrow \mathit{bool}) \Rightarrow \mathit{dfa} \Rightarrow \mathit{dfa} \Rightarrow \mathit{dfa} \\
\mathit{binop\text{-}dfa}\ f\ A\ B \equiv \\
\mathit{let}\ (\mathit{tab}, ps) = \mathit{prod\text{-}dfs}\ A\ B\ (0, 0) \\
\mathit{in}\ (\mathit{map}\ (\lambda(i, j). \\
\qquad \mathit{bdd\text{-}binop}\ (\lambda k\ l.\ \mathit{the}\ \mathit{tab}[k][l])\ (\mathit{fst}\ A)[i]\ (\mathit{fst}\ B)[j]) \\
\quad ps, \\
\quad \mathit{map}\ (\lambda(i, j).\ f\ (\mathit{snd}\ A)[i]\ (\mathit{snd}\ B)[j])\ ps)
\end{array}
$$

Theorem (Correctness)

$$
\begin{array}{l}
\mathit{If}\ \mathit{wf\text{-}dfa}\ A\ n\ \mathit{and}\ \mathit{wf\text{-}dfa}\ B\ n\ \mathit{and}\ \forall\, bs{\in}bss.\ |bs| = n\ \mathit{then} \\
\mathit{dfa\text{-}accepts}\ (\mathit{binop\text{-}dfa}\ f\ A\ B)\ bss = \\
f\ (\mathit{dfa\text{-}accepts}\ A\ bss)\ (\mathit{dfa\text{-}accepts}\ B\ bss).
\end{array}
$$

Quantifiers

Existential quantifiers

- Convert DFA to NFA (trivial)

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- Project away variable(s) to be quantified (yields NFA)

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- Make DFA accept words without trailing zeros (right quotient)

Quantifiers

Existential quantifiers

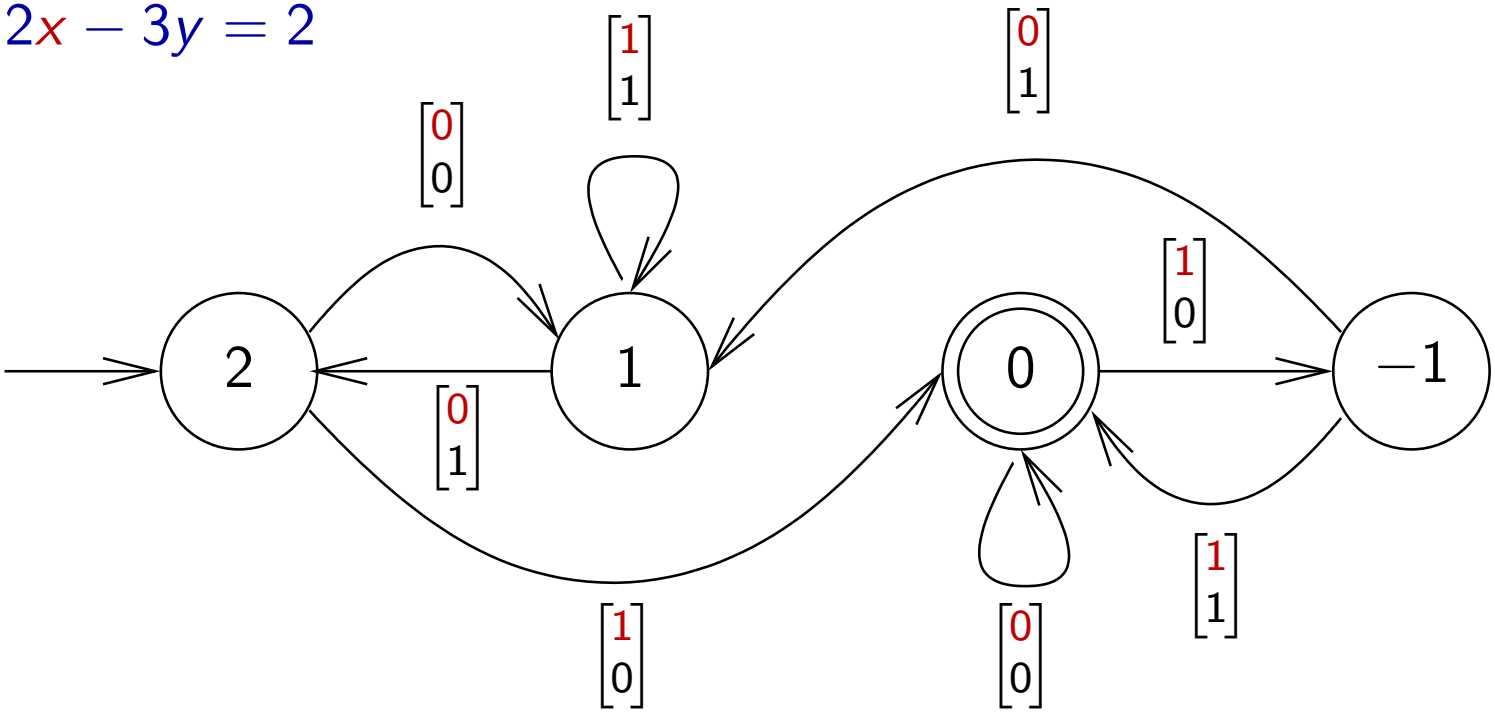
- Convert DFA to NFA (trivial)
- Project away variable(s) to be quantified (yields NFA)
- Convert NFA to DFA (subset construction)
- Make DFA accept words without trailing zeros (right quotient)

Universal quantifiers

Note: $(\forall x.\ P\ x) = (\neg\ (\exists x.\ \neg\ P\ x))$

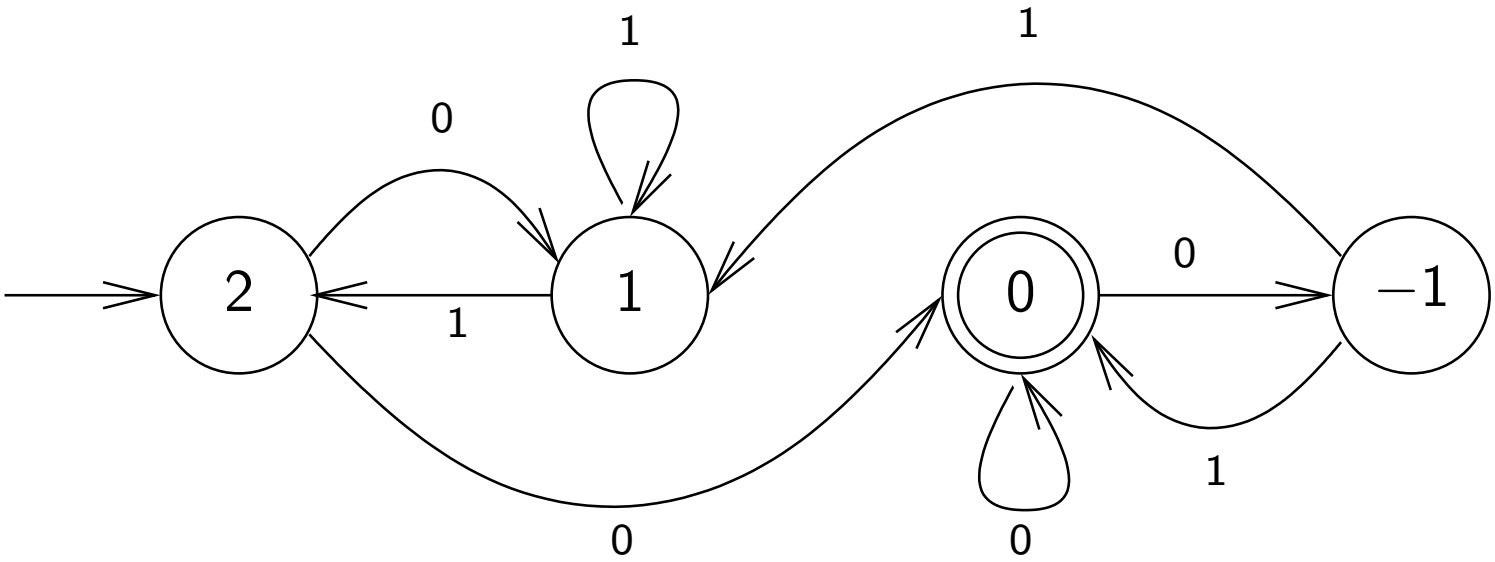
Existential Quantifiers – Example

$2x - 3y = 2$



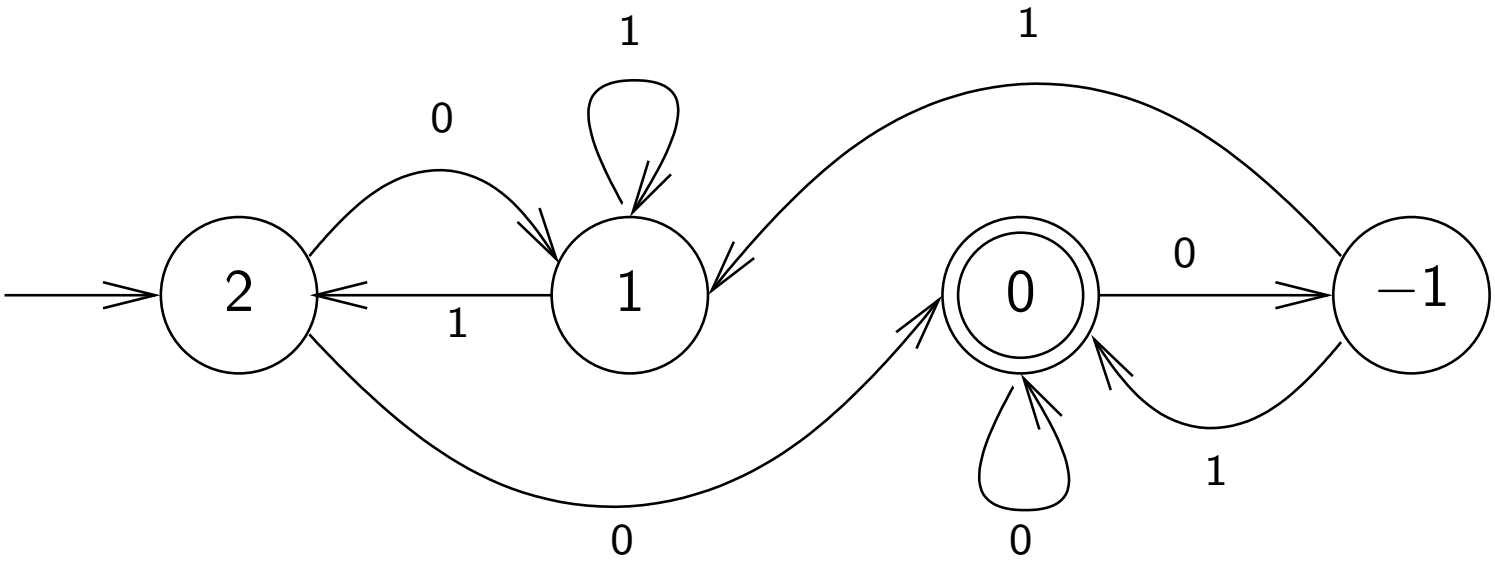
Existential Quantifiers – Example

$\exists x.\ 2x - 3y = 2$ (NFA)

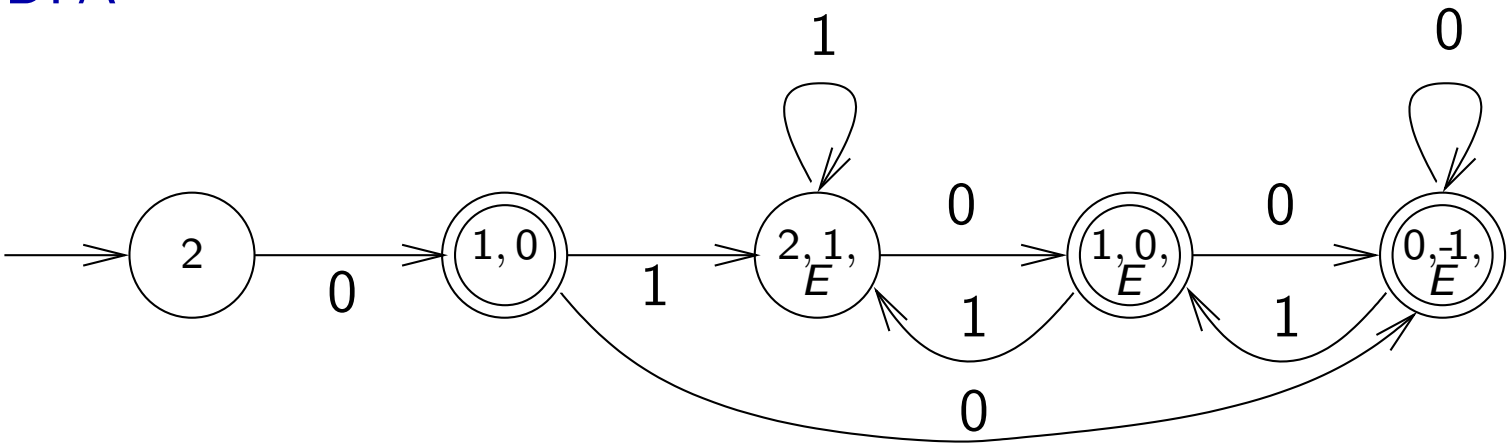


Existential Quantifiers – Example

$\exists x.\ 2x - 3y = 2$ (NFA)



DFA



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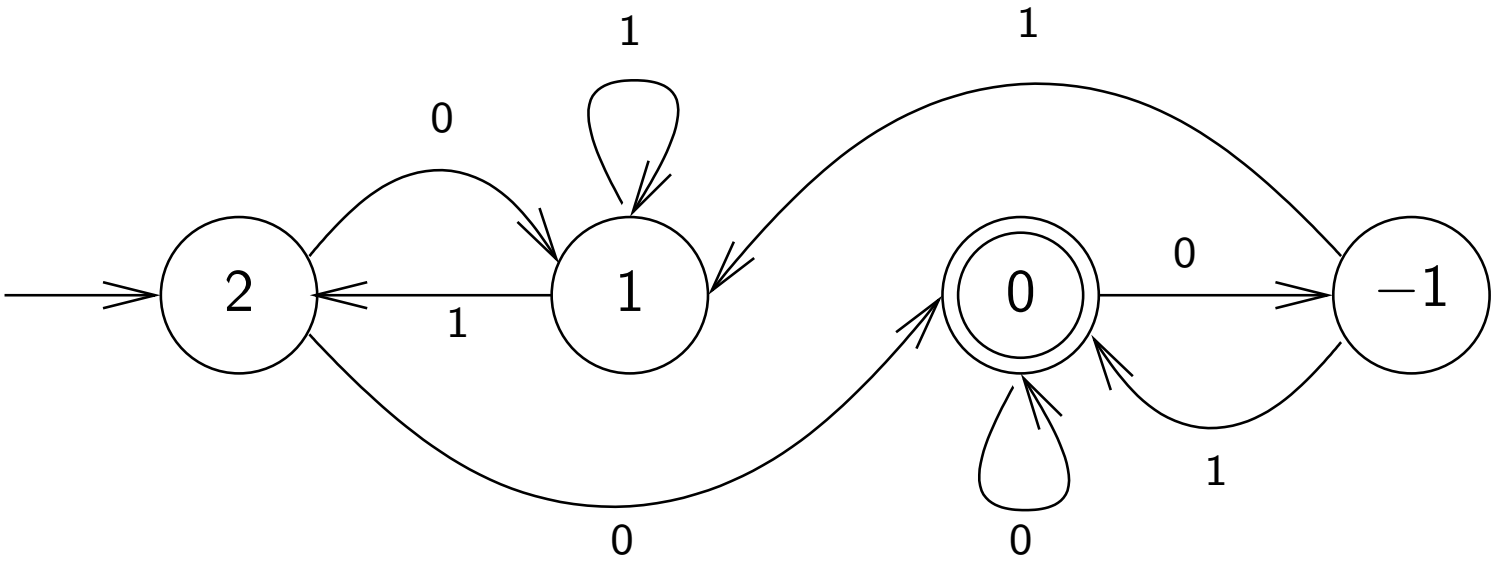
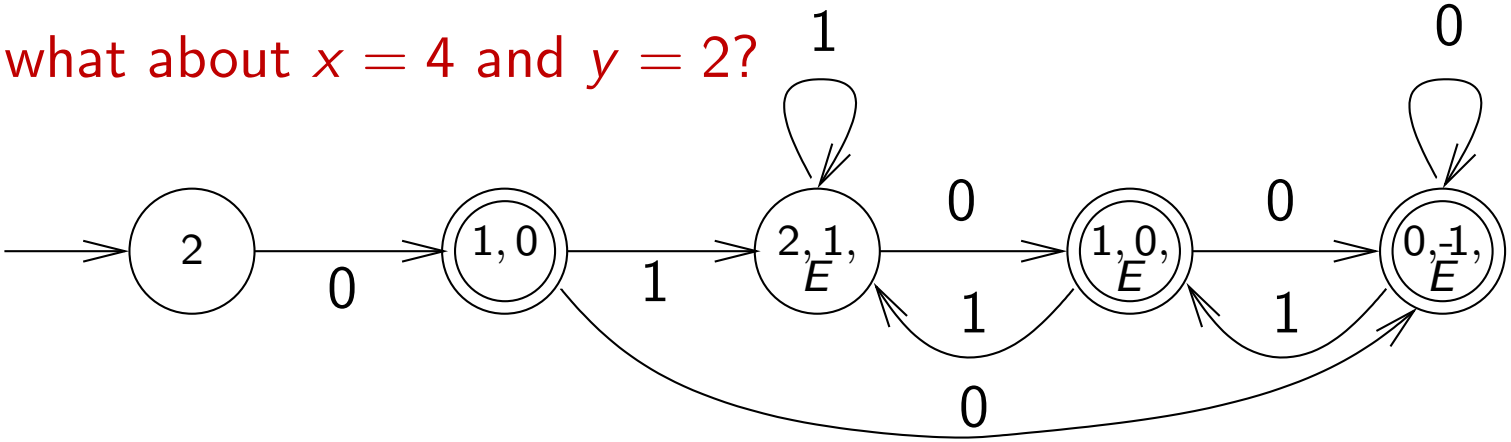


DFA

what about $x = 4$ and $y = 2$?



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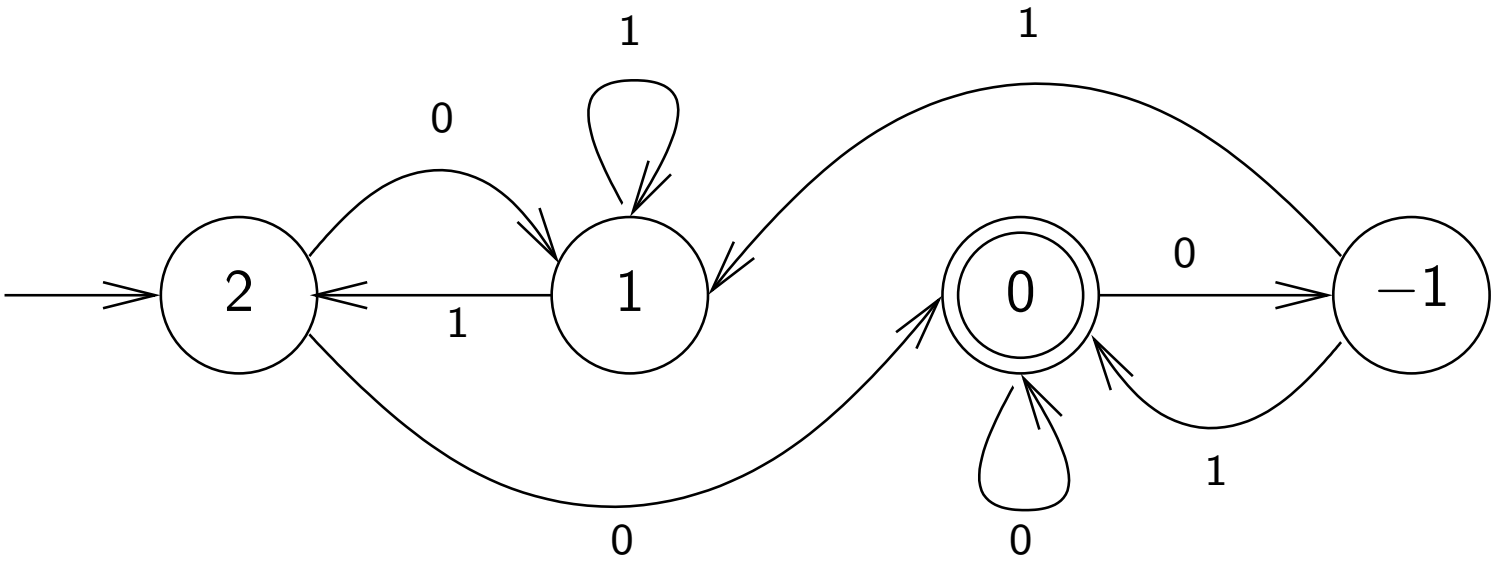
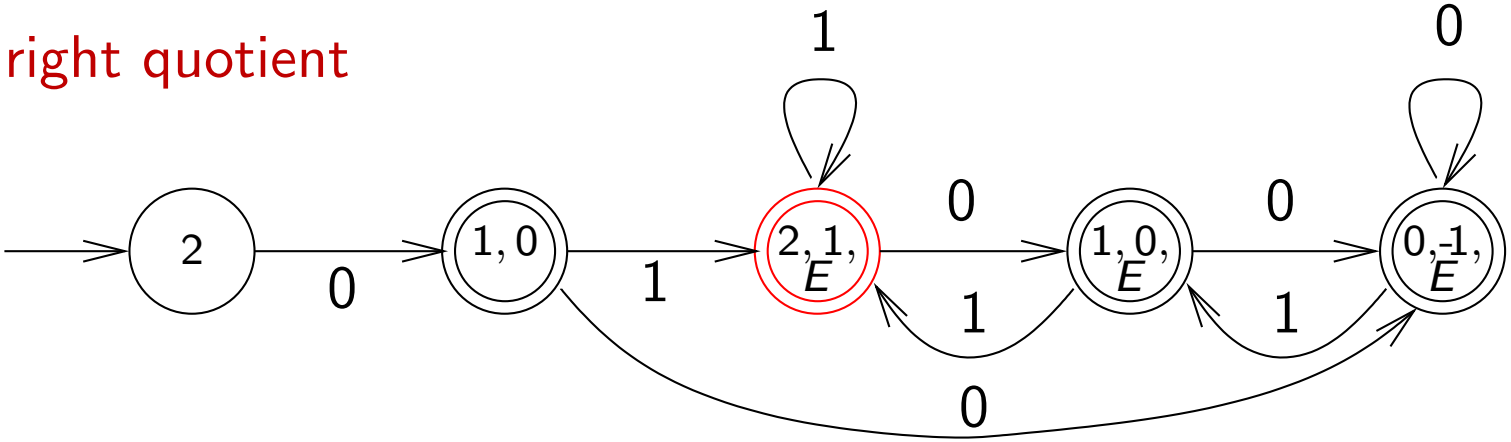
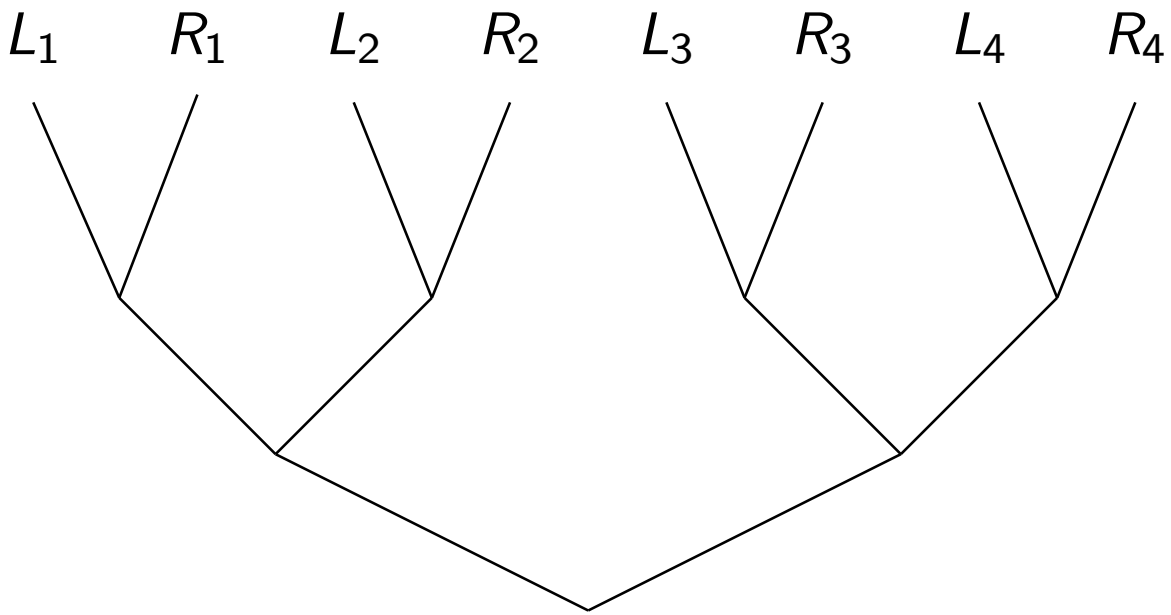


DFA

right quotient



Projection



quantify-bdd :: *nat* ⇒ *bool list bdd* ⇒ *bool list bdd*
quantify-bdd i (*Leaf q*) = *Leaf q*
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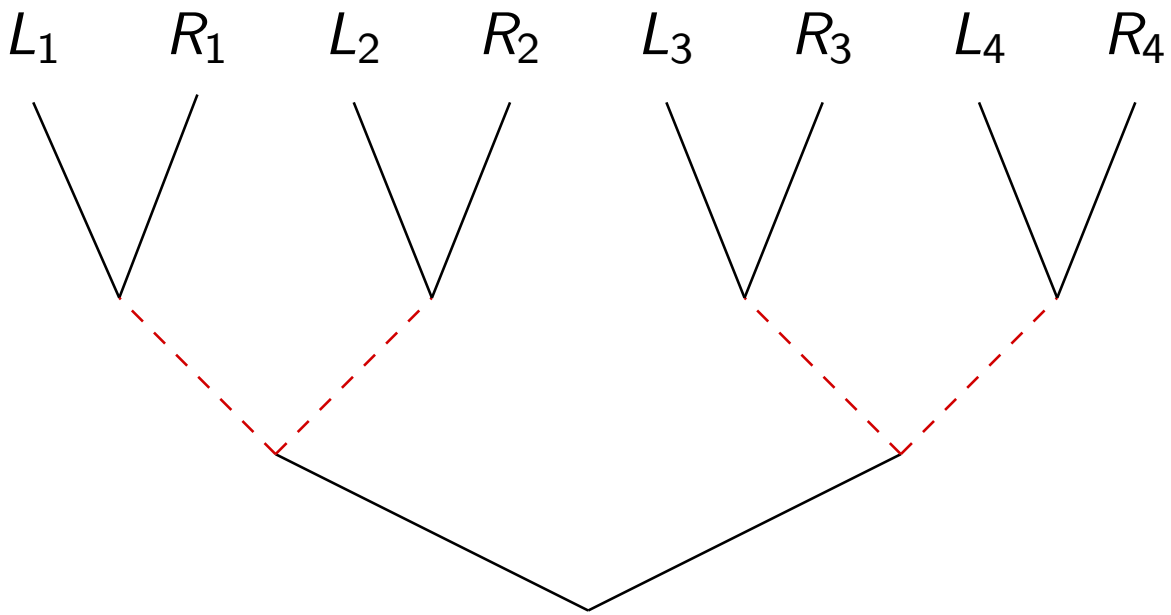
Projection



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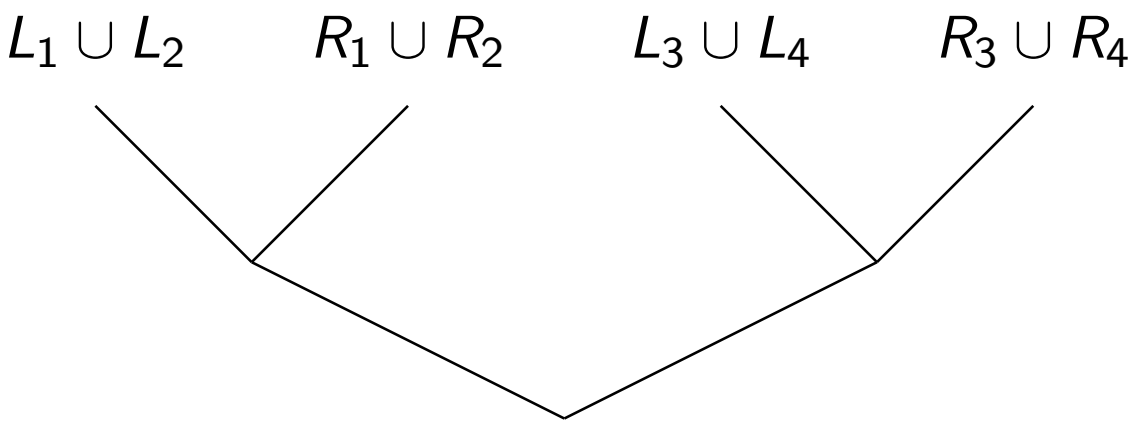

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Inserting Variables

Inserting a row vector into a list of columns

$$insertll :: nat \Rightarrow \alpha\ list \Rightarrow \alpha\ list\ list \Rightarrow \alpha\ list\ list$$

$$insertll\ i\ []\ [] = []$$

$$insertll\ i\ (a \cdot as)\ (bs \cdot bss) = insertl\ i\ a\ bs \cdot insertll\ i\ as\ bss$$

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Inserting an element into a column vector

$$insertl :: nat \Rightarrow \alpha \Rightarrow \alpha\ list \Rightarrow \alpha\ list$$
$$insertl\ i\ a\ [] = [a]$$
$$insertl\ 0\ a\ (b \cdot bs) = a \cdot b \cdot bs$$
$$insertl\ (Suc\ i)\ a\ (b \cdot bs) = b \cdot insertl\ i\ a\ bs$$

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$insertl\ (Suc\ i)\ a\ (b \cdot bs)\ =\ b \cdot insertl\ i\ a\ bs$

Theorem

If $\forall\, bs \in bss.\ |bs| = n$ *and* $|bs| = |bss|$ *and* $i \leq n$ *then*

$nats\text{-}of\text{-}boolss\ (Suc\ n)\ (insertll\ i\ bs\ bss) =$

$insertl\ i\ (nat\text{-}of\text{-}bools\ bs)\ (nats\text{-}of\text{-}boolss\ n\ bss).$

Quantifiers – Correctness

Projection

If wf-nfa A (Suc n) and $i \leq n$ and $\forall bs \in bss.\ |bs| = n$ then
nfa-accepts (quantify-nfa i A) bss =
$(\exists bs.$ nfa-accepts A (insertll i bs bss) $\wedge\ |bs| = |bss|)$.

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DFA of NFA

If wf-nfa A n and $\forall bs \in bss.\ |bs| = n$ then
dfa-accepts (det-nfa A) bss = nfa-accepts A bss.

Quantifiers – Correctness

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DFA of NFA

If wf-nfa A n and $\forall bs \in bss.\ |bs| = n$ then
dfa-accepts (det-nfa A) bss = nfa-accepts A bss.

Right quotient

If wf-dfa A n and $\forall bs \in bss.\ |bs| = n$ then
dfa-accepts (rquot A n) bss =
$(\exists m.$ dfa-accepts A (bss @ zeros m n)).

Outline

The Decision Procedure

```
dfa-of-pf :: nat ⇒ pf ⇒ dfa
dfa-of-pf n (Eq ks l) = eq-dfa n ks l
dfa-of-pf n (Le ks l) = ineq-dfa n ks l
dfa-of-pf n (Neg p) = negate-dfa (dfa-of-pf n p)
dfa-of-pf n (And p q) =
  binop-dfa (∧) (dfa-of-pf n p) (dfa-of-pf n q)
dfa-of-pf n (Or p q) =
  binop-dfa (∨) (dfa-of-pf n p) (dfa-of-pf n q)
dfa-of-pf n (Exist p) =
  rquot (det-nfa (quantify-nfa 0 (nfa-of-dfa (dfa-of-pf (Suc n) p))))
   n
dfa-of-pf n (Forall p) = dfa-of-pf n (Neg (Exist (Neg p)))
```

The Decision Procedure

$$
\begin{aligned}
&\textit{dfa-of-pf} :: \textit{nat} \Rightarrow \textit{pf} \Rightarrow \textit{dfa}\\
&\textit{dfa-of-pf}\ n\ (\textit{Eq}\ \textit{ks}\ l) = \textit{eq-dfa}\ n\ \textit{ks}\ l\\
&\textit{dfa-of-pf}\ n\ (\textit{Le}\ \textit{ks}\ l) = \textit{ineq-dfa}\ n\ \textit{ks}\ l\\
&\textit{dfa-of-pf}\ n\ (\textit{Neg}\ p) = \textit{negate-dfa}\ (\textit{dfa-of-pf}\ n\ p)\\
&\textit{dfa-of-pf}\ n\ (\textit{And}\ p\ q) =\\
&\quad \textit{binop-dfa}\ (\wedge)\ (\textit{dfa-of-pf}\ n\ p)\ (\textit{dfa-of-pf}\ n\ q)\\
&\textit{dfa-of-pf}\ n\ (\textit{Or}\ p\ q) =\\
&\quad \textit{binop-dfa}\ (\vee)\ (\textit{dfa-of-pf}\ n\ p)\ (\textit{dfa-of-pf}\ n\ q)\\
&\textit{dfa-of-pf}\ n\ (\textit{Exist}\ p) =\\
&\quad \textit{rquot}\ (\textit{det-nfa}\ (\textit{quantify-nfa}\ 0\ (\textit{nfa-of-dfa}\ (\textit{dfa-of-pf}\ (\textit{Suc}\ n)\ p))))\\
&\quad\ n\\
&\textit{dfa-of-pf}\ n\ (\textit{Forall}\ p) = \textit{dfa-of-pf}\ n\ (\textit{Neg}\ (\textit{Exist}\ (\textit{Neg}\ p)))
\end{aligned}
$$

Theorem (Correctness)

$$
\begin{aligned}
&\textit{If}\ \forall\, \textit{bs} \in \textit{bss}.\ |\textit{bs}| = n\ \ \textit{then}\\
&\textit{dfa-accepts}\ (\textit{dfa-of-pf}\ n\ p)\ \textit{bss} = \textit{eval-pf}\ p\ (\textit{nats-of-boolss}\ n\ \textit{bss}).
\end{aligned}
$$

Correctness Proof — Existential Quantifier

dfa-accepts (*dfa-of-pf n* (*Exist p*))

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$\longleftrightarrow$ {Definition of *dfa-of-pf*}

dfa-accepts (*rquot* (*det-nfa* (*quantify-nfa 0* (*nfa-of-dfa*
(*dfa-of-pf* (*Suc n*) *p*)))) *n*) *bss*

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$\longleftrightarrow$ {Correctness statement for *rquot*}

$\exists$ *m*. *dfa-accepts* (*det-nfa* (*quantify-nfa 0* (*nfa-of-dfa* (*dfa-of-pf* (*Suc n*) *p*)))) (*bss* @ *zeros m n*)

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$\longleftrightarrow$ {Correctness of *det-nfa*, *quantify-nfa* and *nfa-of-dfa*}

$\exists$ *m bs*. *dfa-accepts* (*dfa-of-pf* (*Suc n*) *p*)
(*insertll 0 bs* (*bss* @ *zeros m n*)) $\wedge$ $|bs| = |bss| + |zeros\ m\ n|$

Correctness Proof — Existential Quantifier

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$\longleftrightarrow$ {Induction hypothesis}

$\exists$ *m bs*. *eval-pf p* (*nats-of-boolss* (*Suc n*)
(*insertll 0 bs* (*bss* @ *zeros m n*))) $\wedge$ $|bs| = |bss| + |zeros\ m\ n|$

Correctness Proof — Existential Quantifier

$$\mathit{dfa\text{-}accepts}\ (\mathit{dfa\text{-}of\text{-}pf}\ n\ (\mathit{Exist}\ p))$$

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$$\longleftrightarrow \quad \{\text{Properties of } \mathit{nats\text{-}of\text{-}boolss}\}$$

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$$\longleftrightarrow \quad \{bs = \mathit{bools\text{-}of\text{-}nat}\ |bss|\ x,\ m = |\mathit{bools\text{-}of\text{-}nat}\ |bss|\ x| - |bss|\}$$

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- Use of DFS pays off!

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- Extend to WS1S, and apply it to circuit verification problems described in [Basin and Friedrich, FROCOS 2000].